Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

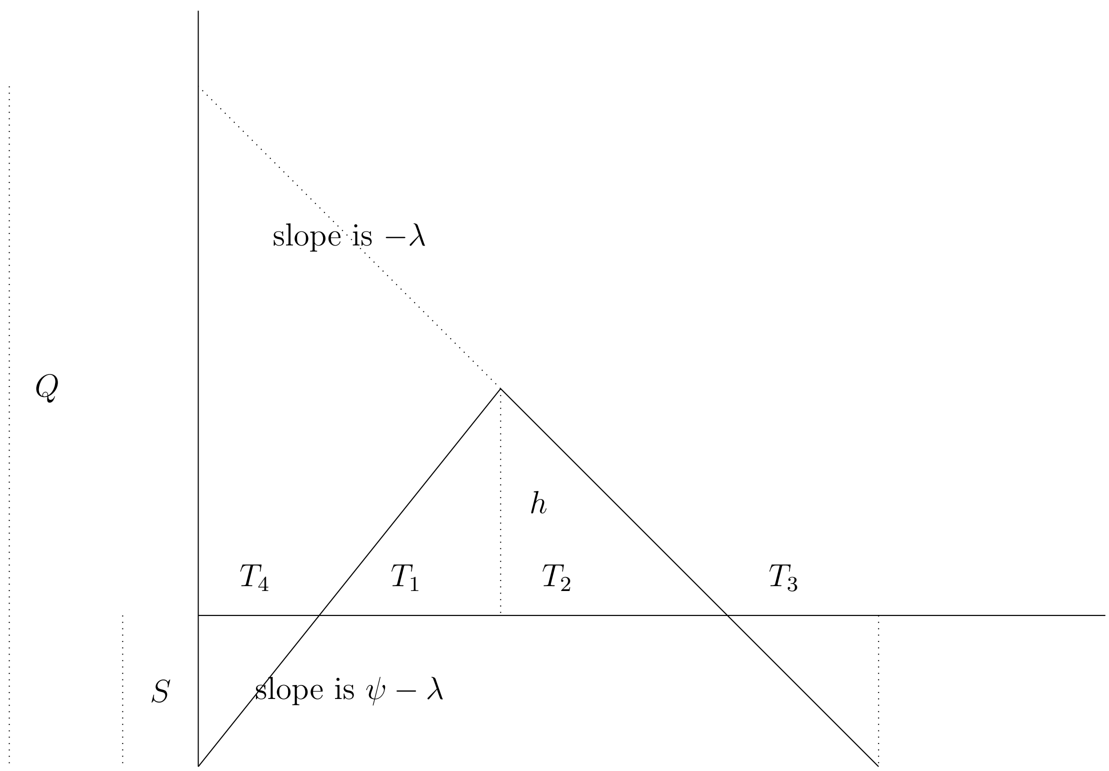
Homework 4: Due Friday October 11.

Q1 In an inventory system:

- A is the fixed cost associated with making an order.
- I is the inventory charge per unit per period.
- π is the back order cost per unit per period.
- λ is the demand per period.
- $\psi > \lambda$ is the rate at which ordered items arrive.

Determine the optimal order strategy and its cost per period.

Solution:



With reference to the diagram above, we have

$$\begin{aligned} Q &= \lambda T \\ T &= T_1 + T_2 + T_3 + T_4 \\ \frac{T_1 + T_2 + T_4}{T} &= \frac{Q - S}{Q}. \\ \frac{T_3}{T} &= \frac{S}{\lambda T} = \frac{S}{Q}. \\ S &= (\psi - \lambda) T_4. \\ h &= (\psi - \lambda) T_1. \\ h &= \lambda T_2. \end{aligned}$$

We deduce from this that

$$\frac{T_1 + T_2}{T} = \frac{Q - S}{Q} - \frac{T_4}{T} = \frac{Q - S}{Q} - \frac{\lambda}{\psi - \lambda} \cdot \frac{S}{Q} = 1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}$$

and

$$\frac{T_3 + T_4}{T} = \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}$$

and

$$T_1 = (T_1 + T_2) \cdot \frac{\lambda}{\psi} = \left(1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}\right) \cdot \frac{Q}{\psi} = \frac{Q}{\psi} - \frac{S}{\psi - \lambda}$$

and then

$$h = Q \cdot \frac{\psi - \lambda}{\psi} - S.$$

The total cost per period is

$$K = \frac{A}{T} + I \cdot \frac{T_1 + T_2}{T} \cdot \frac{h}{2} + \pi \cdot \frac{T_3 + T_4}{T} \cdot \frac{S}{2}.$$

This is good enough. No need to do any more for credit.

We write this in terms of Q, S only.

$$\begin{aligned} \frac{A}{T} &= \frac{A\lambda}{Q}. \\ I \cdot \frac{T_1 + T_2}{T} \cdot \frac{h}{2} &= I \cdot \left(1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}\right) \cdot \left(\frac{Q}{2} \cdot \frac{\psi - \lambda}{\psi} - \frac{S}{2}\right). \\ \pi \cdot \frac{T_3 + T_4}{T} \cdot \frac{S}{2} &= \pi \cdot \left(\frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}\right) \cdot \frac{S}{2}. \end{aligned}$$

We then have

$$\frac{\partial K}{\partial Q} = -\frac{A\lambda}{Q^2} + \frac{I}{2}\left(\frac{\psi-\lambda}{\psi} - \frac{\psi S^2}{(\psi-\lambda)Q^2}\right) - \pi\cdot\frac{\psi}{\psi-\lambda}\cdot\frac{S^2}{2Q^2}.$$
$$\frac{\partial K}{\partial S} = I\cdot\left(\frac{\psi}{\psi-\lambda}\cdot\frac{S}{Q} - 1\right) + \pi\cdot\frac{\psi}{\psi-\lambda}\cdot\frac{S}{Q}.$$

Putting the partial derivatives equal to zero and solving gives us the optimal values

$$Q^* = \left(2A\lambda\cdot\frac{\psi}{\psi-\lambda}\cdot\frac{I+\pi}{I\pi}\right)^{1/2}.$$
$$S^* = \left(2A\lambda\cdot\frac{\psi-\lambda}{\psi}\cdot\frac{I}{\pi(I+\pi)}\right)^{1/2}.$$
$$K^* = \left(2A\lambda\cdot\frac{\psi-\lambda}{\psi}\cdot\frac{I\pi}{I+\pi}\right).$$

Q2 Analyse the following inventory system and derive a strategy for minimising the total cost. There are n products. Product i has demand λ_i per period and no stock-outs are allowed. The cost of making an order for Q units of a mixture of products is AQ^α where $0 < \alpha < 1$. The inventory cost is I times $\max\{L_1, L_2, \ldots, L_n\}$ per period where L_i is the average inventory level of product i in that period.

Solution: We argued in class that we order items at intervals T. Thus we order $Q_i = T\lambda_i$ units of item i each time and $L_i = Q_i/2$. Let $\lambda = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then the inventory cost is therefore $IT\lambda/2$. This gives us a total cost of

$$K = \frac{A\left(T\sum_{j=1}^n \lambda_i\right)^\alpha}{T} + \frac{IT\lambda}{2} = \frac{B}{T^{1-\alpha}} + \frac{IT\lambda}{2}$$

where $B = A\left(\sum_{j=1}^n \lambda_i\right)^\alpha$.

Now

$$\frac{dK}{dT} = -\frac{B(1-\alpha)}{T^{2-\alpha}} + \frac{I\lambda}{2}.$$

Therefore, the optimal value for T is given by

$$T^* = \left(\frac{2B(1-\alpha)}{I\lambda}\right)^{1/(2-\alpha)}.$$

Q3 Give an algorithm to solve the following scheduling problem. There are n jobs labelled $1, 2, \ldots, n$ that have to be processed one at a time on a single machine. There is an acyclic digraph $D = (V, A)$ such that if $(i, j) \in A$ then job j cannot be started until job i has been completed. The problem is to minimise $\max_j f_j(C_j)$ where for all j, f_j is a monotone increasing. As usual, C_j is the completion time of job j. This is distinct from its processing time p_j.

Solution: Let S be the set of jobs with no successor in D i.e. the set of sinks of D. The last job must be in S and it will complete at time $p = p_1 + p_2 + \cdots + p_n$. Let $f_k(p) = \min_{j \in S} f_j(p)$. We schedule k last and then inductively schedule the remaining jobs.