Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 3: Due Monday September 27.

Q1 Can the following shortest path problem be solved by the Dijkstra algorithm? The edges of a digraph are colored Red, Blue and Green. Suppose edge lengths are non-negative, but a path can have at most r Red edges, b Blue edges and no Blue edge can be followed by a Green edge. Give an explicit definition of path length.

Solution Let $\mathcal{R}$ denote the set of restrictions imposed by the colorings. Then the length of a path is given by

$$\ell(P) = \begin{cases} \sum_{e \in P} \ell(e) & P \text{ satisfies } \mathcal{R}. \\ \infty & \text{Otherwise.} \end{cases}$$

Clearly $\ell(P) \geq \ell(Q)$ whenever Q is a subpath of P and so we can use Dijkstra's algorithm.

Q2 Convert the following into a standard assignment problem. We have a bipartite graph with bipartition $A = \{a_1, a_2, \ldots, a_m\}, B = \{b_1, b_2, \ldots, b_n\}$. An assignment now is a set of edges M such (i) a_i is incident to exactly r_i edges of M for $i = 1, 2, \ldots, m$ and (ii) b_j is incident to exactly s_j edges of M for $j = 1, 2, \ldots, n$. Here $\sum_i r_i = \sum_j s_j$. The cost of edge (a_i, b_j) is $c(i, j)$ and the cost of an assignment M is $\sum_{e \in M} c(e)$. The objective is to find a minimum cost assignment.

Solution We replace the vertex a_i by vertices $a_i(1), a_i(2) \ldots, a_i(r_i)$ for $i = 1, 2, \ldots, m$ and the vertex b_j by vertices $b_j(1), b_j(2), \ldots, b_j(s_j)$ for $j = 1, 2, \ldots, n$. The cost of edge $\{a_i(k), b_j(l)\}$ will be $c(a_i, b_j)$. Each solution x to the original problem can be mapped to $\prod_{i=1}^m \prod_{j=1}^n r_i! s_j!$ solutions of the expanded problem, and each of these has the same cost. Each solution to the expanded problem arises from a unique solution to the original problem.

Q3 Let $G = (A, B, E)$ be a bipartite graph. Let $I \subseteq B$ be independent if G contains a matching M that is incident with every vertex in I. Show that the independent sets form a matroid.
Hint: consider the action of augmenting paths.

Solution Clearly the independent sets form an independence system. Next, suppose that I_1, I_2 are independent and that M_1, M_2 are matchings incident with I_1, I_2 respectively and that $|I_1| = |M_1| > |M_2| = |I_2|$. Consider $M_1 \oplus M_2$. It consists of alternating paths and cycles, but because $|M_1| > |M_2|$ there must be at least one alternating path P that goes from a vertex $a \in A$ to a vertex $b \in B$ such that neither a nor b are covered by M_2. Here $b \in I_1 \setminus I_2$. If we amend M_2 by removing $M_2 \cap P$ and adding $M_1 \cap P$ then we will obtain a new matching that covers $I_2 \cup \{b\}$. This verifies the axioms of a matroid.