Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 1: Due Monday September 9.

Describe a Dynamic programming solution to the following problems:

Q1 A company manufactures two products A and B at a certain facility. The demands for the products are $a_r, b_r, r = 1, 2, \ldots, n$ over the next n periods. The cost of making x of either product is $c(x)$ and there is room to store H in total of the two products. Cleaning problems require that only one product can be manufactured in any one period. Assume that at the beginning of period one there is $H/2$ of each product in storage. The problem is to minimise total cost, given that all demands must be met.

Solution Let $f(r, a, b)$ denote the minimum cost of meeting demand in periods $r, r+1, \ldots, n$ given that you start period r with a units of A and b units of B in stock. Then we have the recurrence for $r = 1, 2, \ldots, n$:

$$f(r, a, b) = \min \begin{cases} \min_x \{c(x) + f(r+1, a+x-a_r, b-b_r)\} \\ \min_x \{c(x) + f(r+1, a-a_r, b+x-b_r)\} \end{cases}$$

In addition $f(r, a, b) = \infty$ if $a < 0$ or $b < 0$ or $a + b > H$. Also, $f(n+1, a, b) = 0$ for $a, b \geq 0$.

Q2 You have to drive across country along a road of length L. There are gas stations at points $P_1, P_2, \ldots, P_r$ along the route. Your car can hold g gallons of gasoline. At gas station i, the price of gas is p_i per gallon. If you drive at s miles per hour then you use up $f(s)$ gallons of gas per mile. You can assume that you have to drive at constant speed between stops. You start with a full tank of gas and you have an amount A to spend on the trip. Can you finish the trip in time at most T?
Hint: let $f(i, a, \gamma)$ denote the minimum time to get from P_i to P_r given you are at P_i, you have a left and γ gallons in your car. Find a recurrence for f.

Solution: Let $f(i, a, g)$ be as in the hint. Let $f(r, a, g) = 0$ for all a, g. We have to determine whether or not $f(0, A, g) \leq T$. For this we use the recurrence,

$$f(i, a, \gamma) = \min_{j>i,s,b} \left\{ \frac{P_j - P_i}{s} + f\left(j, a - bp_i, \gamma + b - (p_j - p_i)f(s)\right) \right\}$$

Here j represents the choice of next stop, s represents the speed you will travel and b denotes the amount of gas that you will buy at station i. The constraints are

$$\begin{aligned} \gamma + b - (P_j - P_i)f(s) \geq 0. \\ \gamma + b \leq g. \\ a - bp_i \geq 0. \end{aligned}$$

Q3 The people of a certain area live at the side of a long straight road of length L. The population is clustered into several villages at points $a_1, a_2, \ldots, a_k$ along the road. There is a proposal to build ℓ fire stations on the road. The problem is to build them so that the maximum distance of a village to its nearest fire station is minimised. Formulate the problem of finding the optimum placement of fire stations as a dynamic program. (Assume that fire stations are to be placed at integer points only on the line.)
Hint: for an interval $I = \{i, i+1, \ldots, j\}$ let $d(I, k)$ denote the maximum distance to a fire station placed at k from villages placed in I. Let $D(I) = \min_{k \in I} d(I, k)$. Now break up stick of length L into ℓ intervals $I_1, I_2, \ldots, I_\ell$ and minimise $\max\{D(I_j) :_j= 1, 2, \ldots, \ell\}$.

Solution:

Let $f(x, i)$ be the maximum distance from a village in $[0, x]$ to its nearest fire station in $[0, x]$ if i fire stations are optimally placed to service the villages in $[0, x]$. Then $f(i, i) = 0$ for $i = 0, 1, 2, \ldots, \ell$ and

$$f(x, i) = \min_{i \leq y \leq x} \{\max\{f(y, i-1), d([y, x])\}\}.$$

Here y is the tentative place to put the ith firestation.