Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 2: Due Monday September 24.

Q1 Can the following shortest path problem be solved by the Dijkstra algorithm? The edges of a digraph are colored Red and Blue. Suppose edge lengths are non-negative, but a path can have at most k red edges. Give an explicit definition of path length.

Solution: We let

$$\ell(P) = \begin{cases} \sum_{e \in P} \ell(e) & P \text{ contains at most } k \text{ red edges.} \\ \infty & \text{Otherwise.} \end{cases}$$

Q2 Convert the following 3-dimensional assignment problem into a 2-dimensional problem. There are objects $A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_n\}, C = \{c_1, c_2, \ldots, c_n\}$ and $A \cup B \cup C$ must be partitioned into n triples with one element from each of A, B, C in each triple. The value of a triple $\{a_i, b_j, c_k\}$ is given by $c_{i,j,k} = u_{i,j} + v_{i,k}$. The goal is partition the triples at minimum total cost.
Hint: a partition into triples can be determined by two permutations ϕ, ψ of $[n]$. In which case we have triples $(a_i, b_{\phi(i)}, c_{\psi(i)})$ for $i \in [n]$.

Solution: The objective becomes

$$\sum_{i=1}^{n} (u_{i,\phi(i)} + v_{i,\psi(i)})$$

and we can solve the problem by choosing ϕ to minimise $\sum_{i=1}^{n} u_{i,\phi(i)}$ and ψ to minimise $\sum_{i=1}^{n} v_{i,\psi(i)}$. Both problems are assignment problems.

Q3 Suppose we color the elements of a set A with q colors. Let a subset of S be *rainbow colored* if all of its elements have a different color. Show that the collection of rainbow colored sets forms a matroid.

Solution: If I, J are rainbow and $|J| = |I| + 1$ then J must contain an element e whose color does not appear in I. So, $I \cup \{e\}$ is also rainbow.