Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 1: Due Monday September 10.

Describe a Dynamic programming solution to the following problems:

Q1 An $m \times n$ rectangle of wood is to be cut into smaller rectangles. An $a \times b$ rectangle is worth $m_{a,b}$. The machine that cuts rectangles can only cut full length or full width. I.e. if after cutting there is an $x \times y$ rectangle then the machine can cut it into two rectangles $z \times y$ and $(x - z) \times y$ for some z or into two rectangles $x \times z$ and $x \times (y - z)$. All rectangles cut must have integral side lengths.

Solution Let $f(p, q)$ be the maximum value obtainable from a $p \times q$ rectangle. We can write

$$f(p,q) = \max\{\max_{1\leq x\leq q}\{m_{p,x} + f(p, q - x)\}, \max_{1\leq y\leq p}\{m_{y,q} + f(p - y, q)\}\}.$$

Q2 Consider a 2-D map with a horizontal river passing through its center. There are n cities on the southern bank with x-coordinates $a(1)...a(n)$ and n cities on the northern bank with x-coordinates $b(1)...b(n)$. You want to connect as many north-south pairs of cities as possible with bridges such that no two bridges cross. When connecting cities, you can only connect city i on the northern bank to city i on the southern bank. Construct a Dynamic Programming solution to this problem. (You can assume that $a(1) < a(2) < \cdots < a(n)$, but you **cannot** assume that $b(1) < b(2) < \cdots < b(n)$. If both sequences are increasing, then the problem is trivial).

Solution: Let $f(\ell)$ be the maximum number of bridges we can connect, considering only $a(1), \ldots, a(\ell)$. Then

$$f(\ell) = \max\{f(\ell-1), 1+\max\{f(k) : k < \ell \text{ and bridge } k \text{ avoids bridge } \ell\}\}$$

Q3 Solve the infinite horizon problem for the given matrix of costs. Assume that $\alpha = 1/2$.

$$\begin{bmatrix} 5 & 4 & 1 & 8 \\ 2 & 1 & 5 & 6 \\ 3 & 1 & 5 & 4 \\ 4 & 3 & 6 & 1 \end{bmatrix}$$

Begin with the policy

$$\pi(1) = 4, \pi(2) = 4, \pi(3) = 3, \pi(4) = 4.$$

Solution: We begin by evaluating the solution:

$$\begin{aligned}
y_1 &= 8 + \frac{y_4}{2} = 9. \\
y_2 &= 6 + \frac{y_4}{2} = 7. \\
y_3 &= 5 + \frac{y_3}{2} = 10. \\
y_4 &= 1 + \frac{y_4}{2} = 2.
\end{aligned}$$

We must now check for optimality:

$$\begin{array}{cccc}
5 + 9/2 & 2 + 9/2 & 3 + 9/2 & 4 + 9/2 \\
4 + 7/2 & 1 + 7/2* & 1 + 7/2* & 3 + 7/2 \\
1 + 10/2* & 5 + 10/2 & 5 + 10/2 & 6 + 10/2 \\
8 + 2/2 & 6 + 2/2 & 4 + 2/2 & 1 + 2/2*
\end{array}$$

New policy is

$$\pi(1) = 3, \pi(2) = 2, \pi(3) = 2, \pi(4) = 4.$$