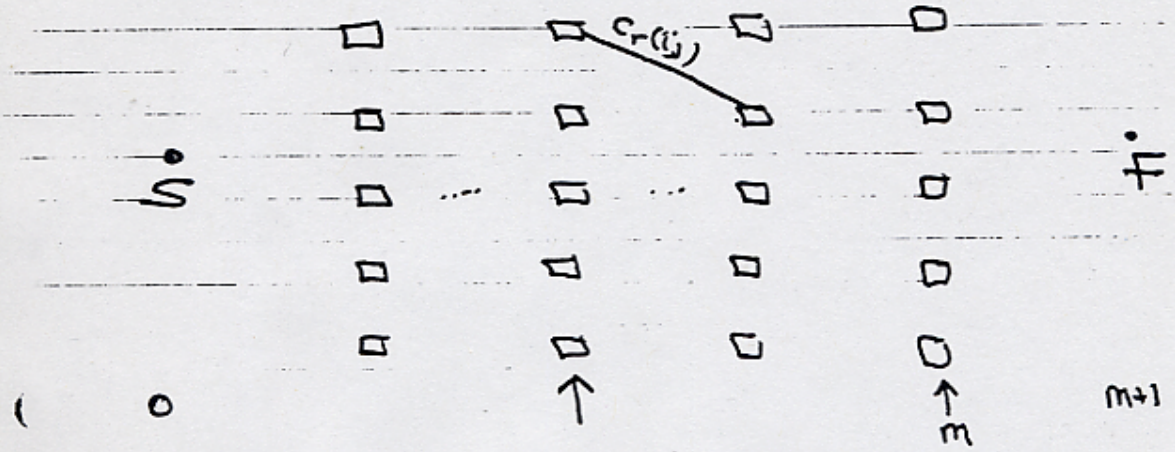
# Dynamic Programming and optimum paths

**Problem**

Find shortest path through mountain ranges




range r

□ represents a pass

$c_r(i,j)$ = length of edge from $i^{th}$ pass in range r to $j^{th}$ pass in range r+1

$f_r(i)$ = shortest distance from $i^{th}$ pass in range r to F

Problem: compute $f_0(S)$

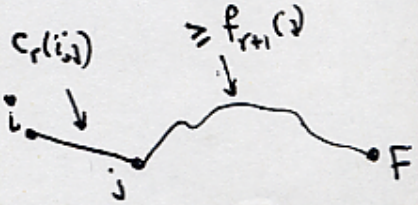
## Functional Equation

$$f_r(i) = \min_j \{ c_r(i,j) + f_{r+1}(j) \} \qquad r = 0, 1, \cdots m \quad (*)$$

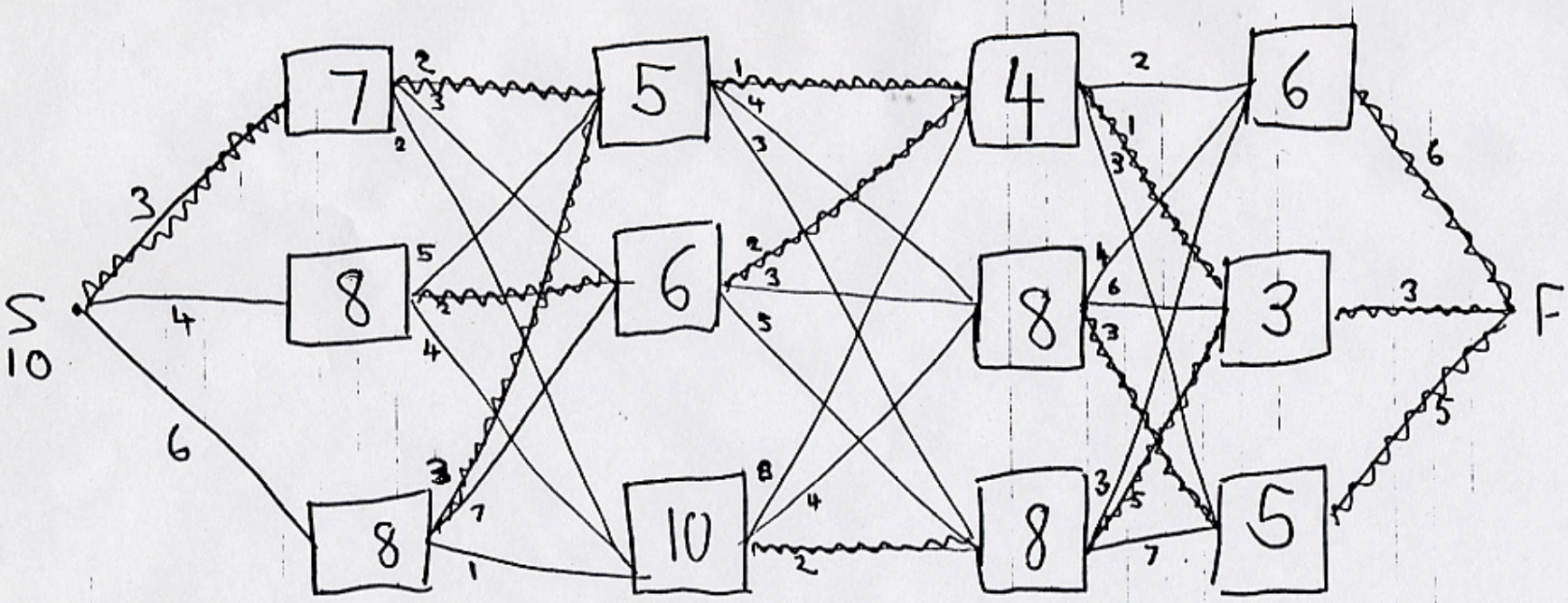
$$f_{m+1}(F) = 0$$

Proof of (*)

$\geq$ RHS : $f_r(i)$ is the length of some path




$\leq$ RHS : can always go from i to F via j which minimizes RHS.




$f_r(i)$

Indicates minimising 'i'

## Dynamic Programming

Dynamic programming is an approach to solving problems rather than a technique for solving a particular problem. The approach can be applied to a wide range of problems, although in many cases it leads to impractical algorithms.

The problem to be tackled is formulated as making a sequence of decisions. Having made one decision, the problem of choosing the remaining decisions is often a similar but 'smaller' version of the original problem. This can lead to a 'functional equation' for finding the best initial decision and each subsequent decision.

### §1 A production problem

As a simple example we consider the following problem: a company estimates the demand $d_j$ for one of its products over the next n periods. It costs the company c(x) to manufacture x units in any one period. All demand must be met in the period in which it occurs but stocks may be built up to provide for demand in future periods. The maximum stock that can be held at any time is H. How much should be produced in each period to minimise the total cost of production. To make the problem self-contained we have to say something about initial and final stocks. Suppose then that there is an initial stock of $i_0$ and that any stock left over at the end of period n is worthless.

The problem then is to decide how much to produce in period 1, how much to produce in period 2 etc. Suppose we decide to

produce an amount $x_1$ in period 1, then at the beginning of period 2 we will have a stock level of $i_0 + x_1 - d_1$ and the problem of minimising the production cost over the next n-1 periods. We can write this down mathematically. Define the quantity $f_r(i)$ to be the minimum cost of meeting demand in periods $r, r+1, \ldots n$ given that one has i units in stock at the beginning of period r.

Focussing temporarily on period 1, we can ask the question, if we decide to produce an amount $x_1$ in period 1, what is the minimum production cost obtainable over the whole n periods? This minimum cost is clearly

$$(1.1) \qquad c_1(x_1) + f_2(i_0 + x_1 - d_1)$$

The first term is the cost of period 1 and the second term in the minimum cost over periods $2,3,\ldots n$ given that we produced $x_1$.

The next question is what is the best value of $x_1$ to take. The answer must be, the value of $x_1$ that minimises (1.1). This will give us the minimum production cost for periods $1,2,\ldots n$ starting with a stock $i_0$ i.e. $f_1(i_0)$. We have thus proved that

$$(1.2) \qquad f_1(i_0) = \min_{x_1} \left(c(x_1) + f_2(i_0 + x_1 - d_1)\right)$$

A similar argument about the decision to be taken at the beginning of period r given that the stock level is currently i shows that in general

(1.3) $$f_r(i) = \min_{x_r} \left(c(x_r) + f_{r+1}\,(i + x_r - d_r)\right)$$

The range over which the 'decision variable' $x_r$ is to be minimised depends on our assumptions about the problem. Firstly we must have $x_r \geq 0$ and since we must produce enough to meet the demand $d_r$, we must have $i + x_r \geq d_r$. The maximum stock level is H and consequently we must have $i + x_r - d_r \leq H$. Thus $x_r$ is to be chosen in the range

(1.4) $$\max (0, d_r - i) \leq x_r \leq H + d_r - i.$$

Now the argument that produced (1.3) only read holds true for $r \leq n-1$, basically because we have not defined $f_{n+1}(i)$. Examining our assumption about final stocks we can see that this is equivalent to

(1.5) $$f_n(i) = \min_{x_n} \left(c(x_n)\right)$$

This can be put into the framework of (1.3) by defining $f_{n+1}(i) = 0$. Equations 1.3 and 1.5 give us a means of solving our problem. We first calculate $f_n(i)$ for $i = 0,1,2,\ldots H$. We thsn use (1.3) to calculate $f_{n-1}(i)$ for $i = 0,1,2,\ldots H$, and then $f_{n-2}(i)$ and so on until we reach $f_1(i)$. If the production quantities x need not be integral then we have to approximate by dividing the range $[0,H]$ into a suitable number of points - depending on the accuracy required and computer storage and time available.

Let us solve the above problem when $n = 4$, $d_j = 3$ in all periods, the maximum stock level $H = 4$ and $c(x) = 18x - x^2$.

So that we can keep track of the optimal production policy we make a note of the value of $x_r$ minimising the R.H.S of (1.3) for each i. Denote this value by $x_r(i)$.

Stage 1 - calculation of $f_4$

By definition $f_4(i) = \min (18x-x^2 | \max(0,3-i) \leq x \leq 7-i)$

$f_4(0) = 45$, $x_4(0) = 3$; $f_4(1) = 32$, $x_4(1) = 2$; $f_4(2) = 17$, $x_4(2) = 1$; $f_4(3) = 0$, $x_4(3) = 0$; $f_4(4) = 0$, $x_4(4) = 0$

Stage 2 - calculation of $f_3$

In this case 1.3 becomes

$$f_3(i) = \min (18x-x^2 + f_4(i + x - 3) | \max(0,3 - i) \leq x \leq 7 - i)$$

$$f_3(0) = \min(45 + f_4(0), 56 + f_4(1), 65 + f_4(2), 72 + f_4(3), 77 + f_4(4)) = 72$$

and $x_3(0) = 6$

Continuing this we build up the table

| i | $f_4(i)$ | $x_4(i)$ | $f_3(i)$ | $x_3(i)$ | $f_2(i)$ | $x_2(i)$ | $f_1(i)$ | $x_1(i)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 45 | 3 | 72 | 6 | 109 | 7 | 142 | 7 |
| 1 | 32 | 2 | 65 | 5 | 104 | 216 | 135 | 5/6 |
| 2 | 17 | 1 | 56 | 4 | 89 | 1 | 126 | 1 |
| 3 | 0 | 0 | 45 | 0/3 | 72 | 0 | 109 | 0 |
| 4 | 0 | 0 | 32 | 0/2 | 65 | 0 | 104 | 0/2 |

Suppose for example that the initial stock level in period 1 is 0. We see from the table that the minimum total production cost is 142. The optimal production policy is found as follows:

$x_1(0) = 7$ i.e. given a stock level of 0 at the beginning of period 1 the optimum production for period 1 is 7. Producing 7 in period 1 means we start period 2 with a stock level 4. From the table $x_2(4) = 0$ i.e. given a stock level of 4 at the beginning of period 2 the optimum production for period 2 is 0. This means we start period 3 with stock level 1. Now $x_3(1) = 5$, so we produce 5 units in period 3 and therefore start period 4 with initial stock 3. As $x_4(3) = 0$ we produce nothing in this period. Thus the optimal policy starting period 1 with zero stock is

| $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|
| 7 | 0 | 5 | 0 |

We may in a similar manner use the table to find the optimum policy for all possible initial stock levels.

In the method above we have worked backwards from period n in calculating the optimum policy. This is called the backward formulation of the problem.

It is also possible to solve the problem working forwards from period 1, giving us a forward formulation.

In the backward formulation model we nad to be explicit on what happened to the final stock, in the forward formulation we have to fix the initial stock at some value. For simplicity assume the initial stock is zero.

Now let us define the quantity $g_r(i)$ to be the minimum cost of meeting demand in periods 1,2,...r given that the stock level at the <u>end</u> of period r is i. Then arguing in a similar manner to

the backward formulation we get

$$(1.6) \qquad g_1(i) = c(i + d_1)$$

$$(1.7) \qquad g_r(i) = \min_{x_r}(c(x_r) + g_{r-1}(i + d_r - x_r))$$

where $x_r$ in 1.7 ranges over

$$\max(0, i + d_r - H) \leq x_r \leq i + d_r$$

Starting with $g_1$ as defined in (1.6) we use (1.7) iteratively to calculate $g_n$ and we can thus calculate an optimum for any value of the final stock.

(i) Add a holding cost

(ii) ~~Add~~ Allow backordering, up to $-B$

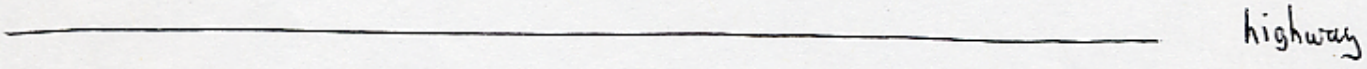
(iii) Add a "smoothing" penalty.

## Problem

A stick of length $L$ is to be broken into pieces of integer length. Let $v_{i,j}$ be the "value" of a piece $[i, i+1, \ldots j]$.

How should the stick be broken in order to maximise the total value.

## Example




$v_{i,j}$ = franchise value of stretch $i,j$ for some enterprise

## Solution

Let $f(r)$, $r = 0, 1, 2, \cdots L$ be maximum value obtainable from a stick of length $r$.

$$f(0) = 0$$

$$f(r) = \max_{0 \le i < r} \{ f(i) + v_{i,r} \} \qquad 0 < r \le L$$

So $f(L)$ can be computed in $O(L^2)$ operations.

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Suppose next that stick must be broken into $k$ pieces. Now use $f(j,r)$, $j = 1, 2, \cdots k$, $r = 0, 1, \cdots L$.

$$f(j,r) = 0 \qquad j > r$$

$$= \max_{0 \le i < r} [ f(j-1, i) + v_{i,r} ]$$

So $f(k,L)$ can be computed in $O(kL^2)$ operations.

A problem with an infinite time horizon

A *system* can be in one of a set $V$ of possible states. For each $v \in V$ one can choose any $w \in V$ and move to $w$ at a cost of $c(v,w)$. The system is to run *forever* and it is requiredto minimise the *discounted cost* of running the system, assuming that the discount factor is $\alpha$. A *policy* is a function $\pi : V \to V$. So if $|V| = n$ then there are $n^n$ distinct policies to choose from.

**Example**

$$\text{Costs} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 2 \\ 1 & 3 & 2 \end{bmatrix} \quad \alpha = 1/2.$$

Let $\pi$ be a policy and let $y_v$ be the discounted cost of this policy, starting at $v \in V$. Then

$$y_v = c(v, \pi(w)) + \alpha y_{\pi(v)} \qquad v \in V. \tag{1}$$

**Example** Let $\pi(1) = \pi(2) = \pi(3) = 1$. Then

$$\begin{aligned} y_1 &= 2 + \frac{1}{2} y_1 \\ y_2 &= 4 + \frac{1}{2} y_1 \\ y_3 &= 1 + \frac{1}{2} y_1. \end{aligned}$$

So

$$y_1 = 4,\ y_2 = 6,\ y_3 = 3.$$

Problem: Find the policy $\pi^*$ which minimises $y_v$ simultaneously for all $v \in V$.

**Theorem 1 Optimality Criterion**
*$\pi^*$ is optimal iff its values $y_v^*$ satisfy*

$$y_v^* = \min_{w \in V} \{c(v,w) + \alpha y_w^*\} \qquad \forall v \in V. \tag{2}$$

**Proof** Suppose that (2) does not hold for some $\pi$.

$$\begin{aligned} y_u &> c(u, \lambda(u)) + \alpha y_{\lambda(u)} \qquad u \in U \\ y_v &= \min_{w \in V} \{c(v,w) + \alpha y_w\} \qquad u \notin U \end{aligned}$$

Define $\tilde{\pi}$ by $\tilde{\pi}(u) = \lambda(u)$ for $u \in U$ and $\tilde{\pi}(v) = \pi(v)$ for $v \notin U$. Then for $u \in U$,

$$\begin{aligned} y_u &> c(u, \lambda(u)) + \alpha y_{\lambda(u)} \\ \tilde{y}_u &= c(u, \lambda(u)) + \alpha \tilde{y}_{\lambda(u)} \end{aligned}$$

So if $\xi_v = y_v - \tilde{y}_v$ for $v \in V$ then

$$\xi_u > \alpha \xi_{\tilde{\pi}(u)} \qquad u \in U. \tag{3}$$

Also, for $v \notin U$

$$\begin{aligned} y_v &= c(v, \pi(v)) + \alpha y_{\pi(v)} \\ \tilde{y}_v &= c(v, \pi(v)) + \alpha \tilde{y}_{\pi(v)} \end{aligned}$$

and so

$$\xi_v = \alpha \xi_{\tilde{\pi}(v)} \qquad v \notin U. \tag{4}$$

It follows from (3), (4) that

$$\begin{aligned} \xi_v &\geq \alpha^t \xi_{\tilde{\pi}^t(v)} \qquad \forall v \notin U, t \geq 1 \\ \xi_u &> \alpha^t \xi_{\tilde{\pi}^t(u)} \qquad \forall u \in U, t \geq 1 \end{aligned}$$

Letting $t \to \infty$ we see that

$$\xi_v \geq 0 \ \forall v \text{ and } \xi_u > 0 \ \forall u \in U.$$

Thus $\tilde{\pi}$ is *strictly better* than $\Pi$ i.e. if (2) does not hald, then we can improve the current policy.
Conversely, if (2) holds and $\hat{\pi}$ is any other policy and $\eta_v = \hat{y}_v - y_v^*$ then

$$\begin{aligned} \hat{y}_v &= c(v, \hat{\pi}(v)) + \alpha \hat{y}_{\hat{\pi}(v)} \\ y_v^* &\leq c(v, \hat{\pi}(v)) + \alpha y^*_{\hat{\pi}(v)} \end{aligned}$$

and so

$$\eta_v \geq \alpha \eta_{\hat{\pi}(v)} \geq \cdots \geq \alpha^t \eta_{\hat{\pi}^t(v)} \qquad \text{for } t \geq 1$$

which implies that $\eta_v \geq 0$ for $v \in V$.
**Policy Improvement Algorithm**

1. Choose arbitrary initial policy $\pi$.
2. Compute $y$ as in (1).
3. If (2) holds – current $\pi$ is optimal, stop.
4. If (2) doesn't hold then
5. compute $\lambda$ by
   $y_{\lambda(v)} = \min_w \{c(v, w) + \alpha y_w\}$.
6. $\pi \leftarrow \lambda$.
7. goto 2.

In our example with $\pi = (1, 1, 1)$. First compute $\lambda = (1, 3, 1)$. Re-compute $y = (\frac{39}{28}, \frac{11}{14}, \frac{95}{56})$. Now $\lambda = \pi$ i.e. (1) holds and we are done.

**Traveling SalesPerson via Dynamic programming:**
We are given a matrix of costs $c(i,j), 1 \leq i,j \leq n$. The problem is to find a permutation $\pi$ of $[n] = \{1,2,\ldots,n\}$ that minimises

$$TSP(\pi) = c_{1,\pi(1)} + c(\pi(1),\pi^2(1)) + \cdot + c(\pi^n)(1),1).$$

This represents the total cost of a "tour through $[n]$ in the order $1,\pi(1),\pi^2(1),\ldots,\pi^n(1),1$. There are $(n-1)!$ distinct tours (each tour, as a set of directed edges of $\vec{K}_n$, arises from $n$ distinct permutations.)
With DP we can solve the problem in $O(n^2 2^n)$ time. For $1 \in S \subseteq [n]$ and $x \in S$, let $f(x,S)$ denote the minimum cost of a path that begins at 1, ends at $x$ and visits each vertex in $S$ exactly once. Then, $f(x,S) = 0$ for $S = \{1\}$ and

$$f(x,S) = \min\{f(x,S \setminus \{x\}) + c(z,x) : z \in S \setminus \{x\}\}.$$

There are $\binom{n-1}{k-1}$ choices for $|S| = k$ and given $S$ there are $k-1$ choices for $x$ and then $k-2$ choices for $y$. So, to compute $f(x,[n])$ for all $1 \neq x \in [n]$ takes time

$$\sum_{k=2}^{n}(k-1)(k-2)\binom{n-1}{k-1} = \sum_{k=3}^{n}(k-1)(k-2)\binom{n-1}{k-1} =$$
$$(n-1)(n-2)\sum_{k=3}^{n}\binom{n-3}{k-3} = (n-1)(n-2)2^{n-3}.$$

To finish we compute $\min\{f(x,[n]) + c(x,1) : x \neq 1)$.