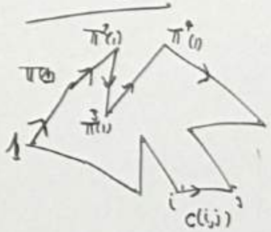
TSP

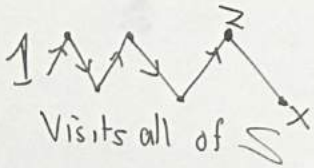


Tour: $1 \to \pi(1) \to \pi^2(1) \to \cdots \to \pi^{n-1}(1) \to 1$

π is a permutation.

solutions $= (n-1)!$

D.P. solves problem in $O(n^2 2^n)$

$f(x,S)$ = min. length path 1  x Visits all of S

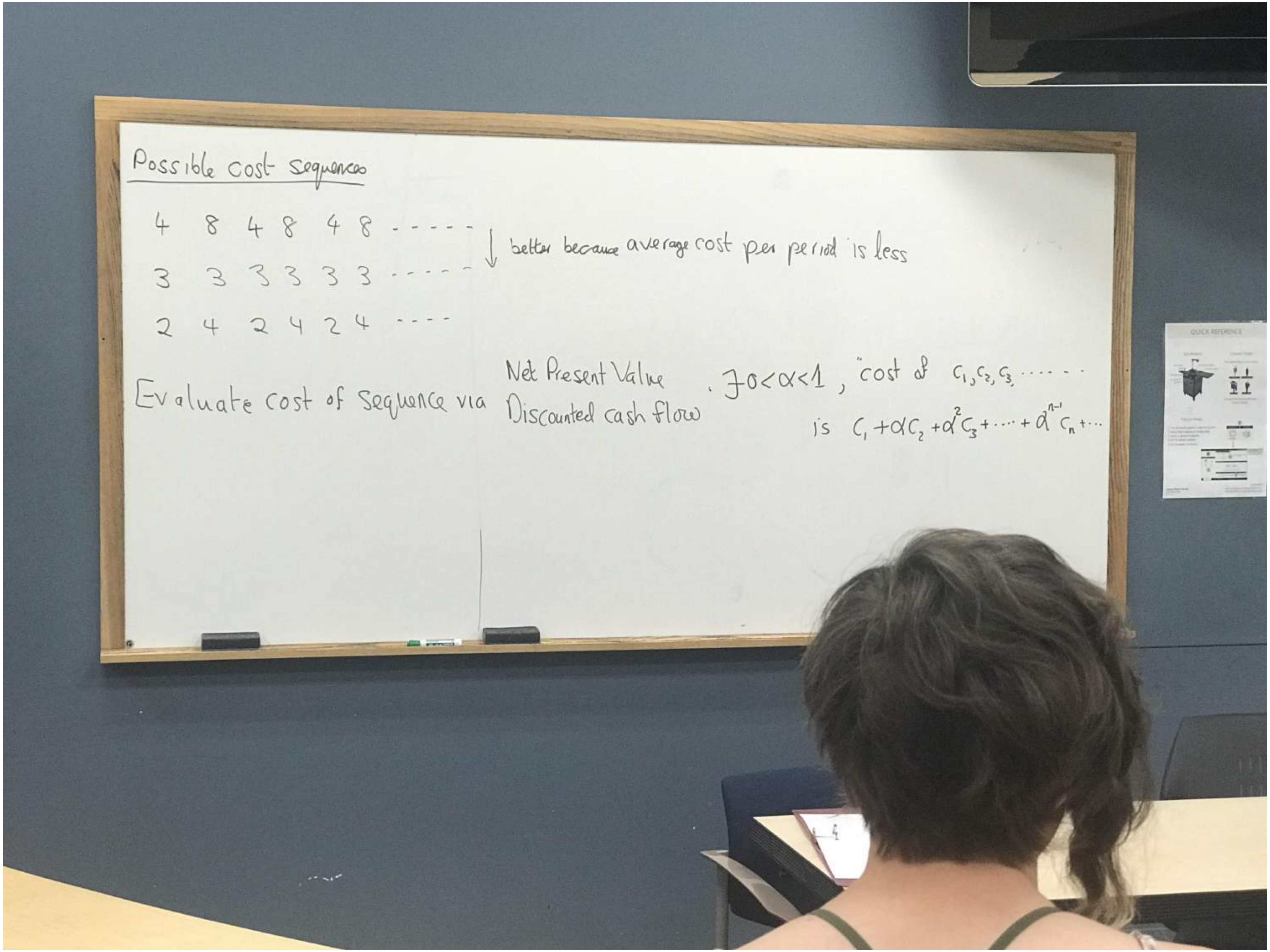
$1 \in S \subseteq [n]$
$x \in S$

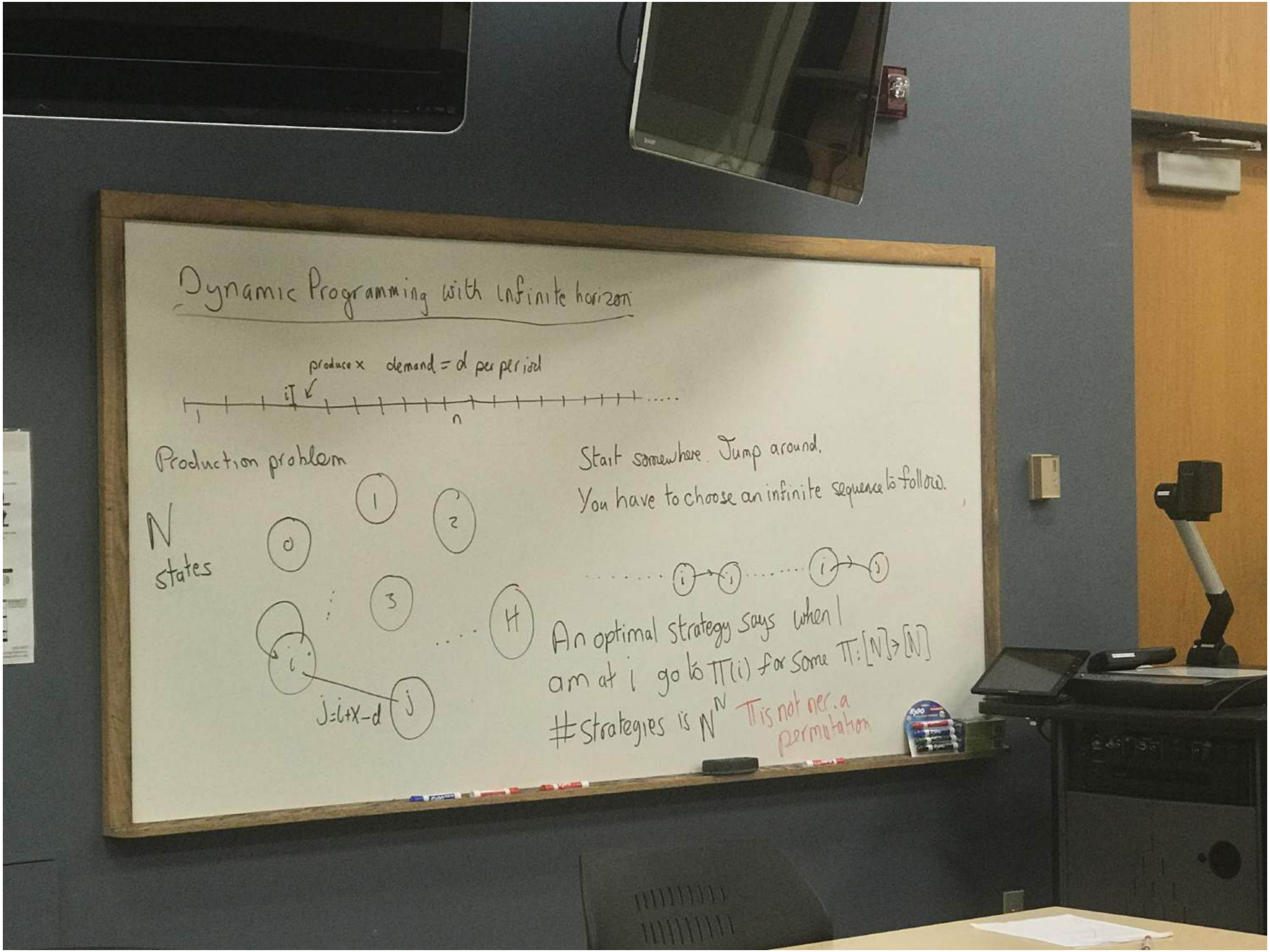
$$= \min_{\substack{z \in S \\ z \neq x,1}} c(z,x) + f(z, S \setminus \{x\})$$

# of choices for x, S, $|S| = k$ is $\underset{\text{Choose } S}{\binom{n-1}{k-1}} \times \underset{\text{Choice } x}{(k-1)} \times \underset{\text{Choice } z}{(k-2)}$

$$\sum_{k=3}^{n} (k-1)(k-2)\binom{n-1}{k-1} = \sum_{k=3}^{n} (n-1)(n-2)\binom{n-3}{k-3} = (n-1)(n-2)\underbrace{\sum_{k-3=0}^{n-3}\binom{n-3}{k-3}}_{2^{n-3}}$$

Min length tour $= \min_x f(x,[n]) + c(x,1)$

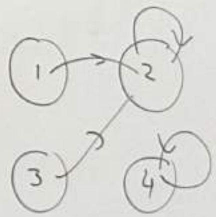

Possible cost sequences
4 8 4 8 4 8 - - - - -
better because average cost per period is less
3 3 3 3 3 3 - - - - -
2 4 2 4 2 4 - - - -
Evaluate cost of sequence via
Net Present Value
Discounted cash flow
$\exists\, 0<\alpha<1$, "cost of $c_1, c_2, c_3, \ldots\ldots$
is $c_1 + \alpha c_2 + \alpha^2 c_3 + \cdots + \alpha^{n-1} c_n + \cdots$


Dynamic Programming with infinite horizon
produce x
demand = d per period
i
1
n
Production problem
N states
0
1
2
3
H
i
j
j=i+x-d
Start somewhere. Jump around.
You have to choose an infinite sequence to follow.
i
j
An optimal strategy says when I
am at i go to π(i) for some π:[N]→[N]
#strategies is N^N
π is not nec. a permutation

Costs $\begin{bmatrix} 3 & 4 & 3 & 2 \\ 5 & 6 & 1 & 4 \\ 7 & 5 & 3 & 2 \\ 6 & 4 & 4 & 7 \end{bmatrix}$ $\alpha = \frac{1}{2}$

Initial π:

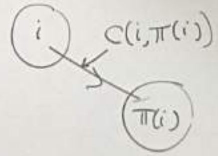
i	1	2	3	4
$\pi(i)$	2	2	2	4



Evaluate Strategy π.

$y_\pi(i)$ = discounted cost, starting from state i at time 0.

$$y_\pi(i) = C(i, \pi(i)) + \alpha y_\pi(\pi(i))$$



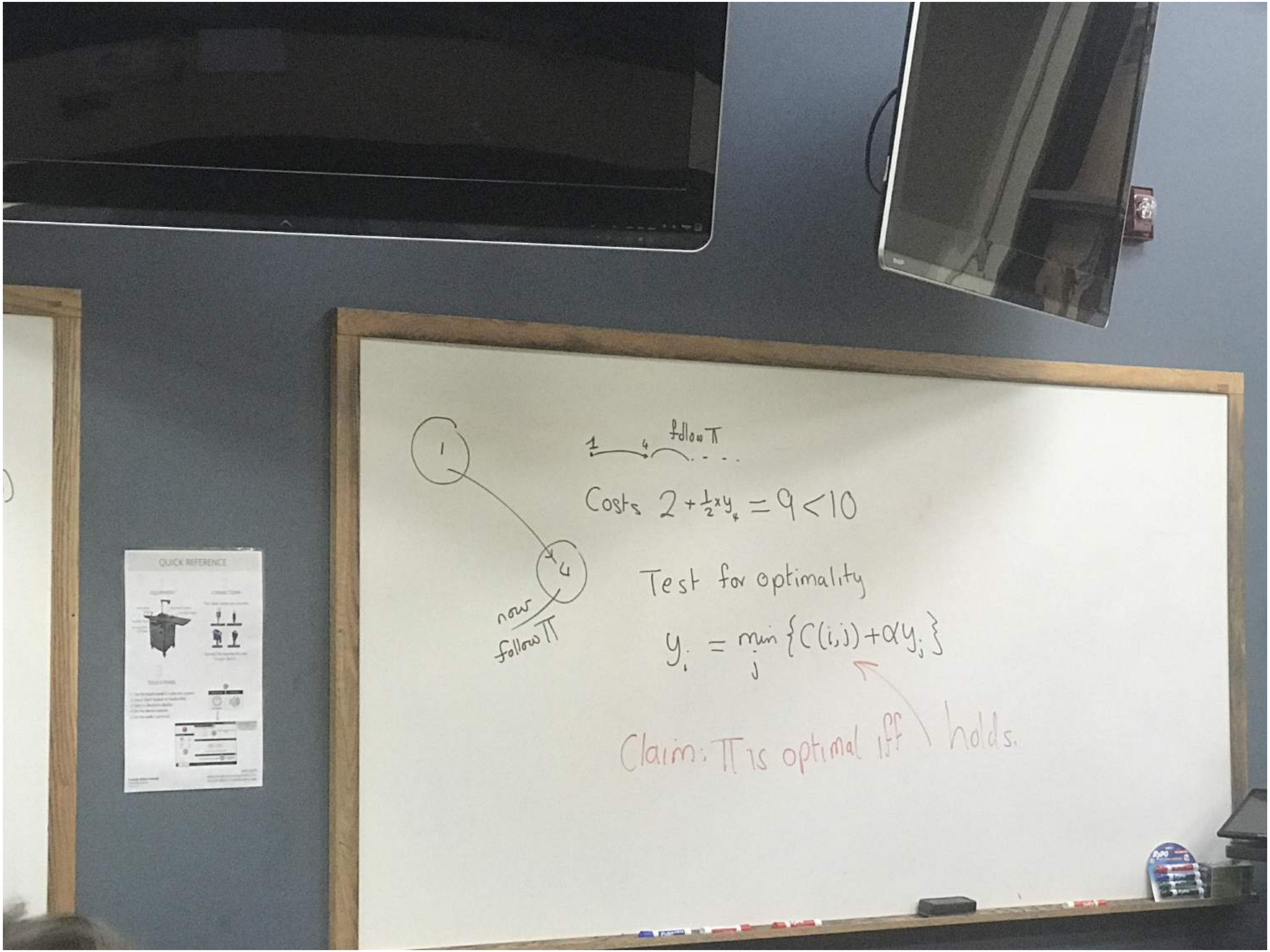
Example

$y_1 = 4 + \frac{1}{2} \times y_2 = 10$

$y_2 = 6 + \frac{1}{2} y_2 = 12$

$y_3 = 5 + \frac{1}{2} y_2 = 11$

$y_4 = 7 + \frac{1}{2} y_4 = 14$


QUICK REFERENCE
1
4
now follow π
follow π
$1 \longrightarrow 4 \frown \cdots$
Costs $2 + \frac{1}{2} \times y_4 = 9 < 10$
Test for optimality
$y_i = \min_j \{ C(i,j) + \alpha y_j \}$
Claim: π is optimal iff holds.

$$\begin{bmatrix} 3 & 4 & 3 & 2 \\ 5 & 6 & 1 & 4 \\ 7 & 5 & 3 & 2 \\ 6 & 4 & 4 & 7 \end{bmatrix} \qquad \alpha = \frac{1}{2}$$

$y_1 = 10$

$y_2 = 12$

$y_3 = 11$

$y_4 = 14$

i	$\pi(i)$
1	2
2	2
3	2
4	4

Optimality Condition

$$y_i = \min_j \; c(i,j) + \alpha y_j \qquad (*)$$

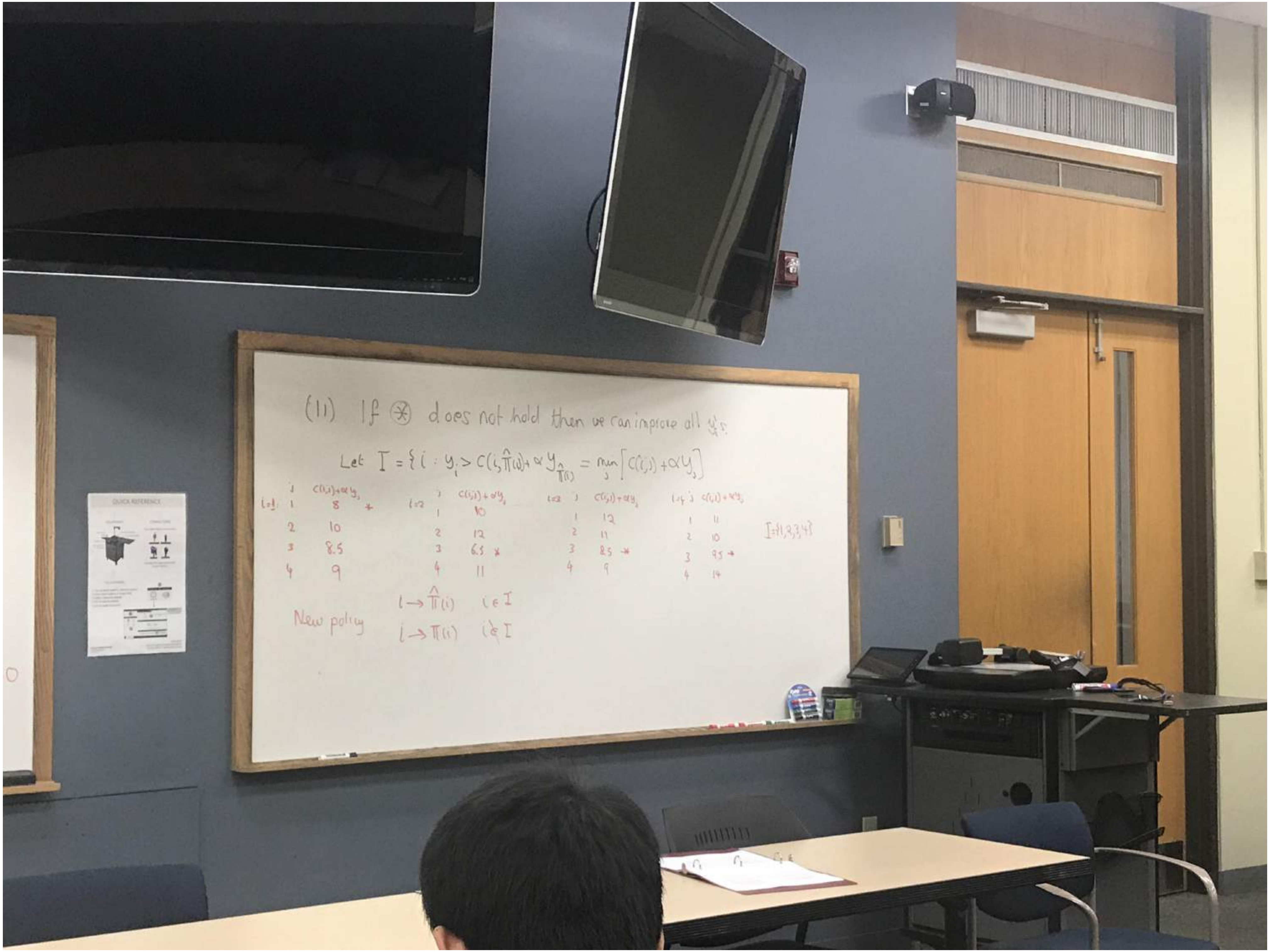
Claim

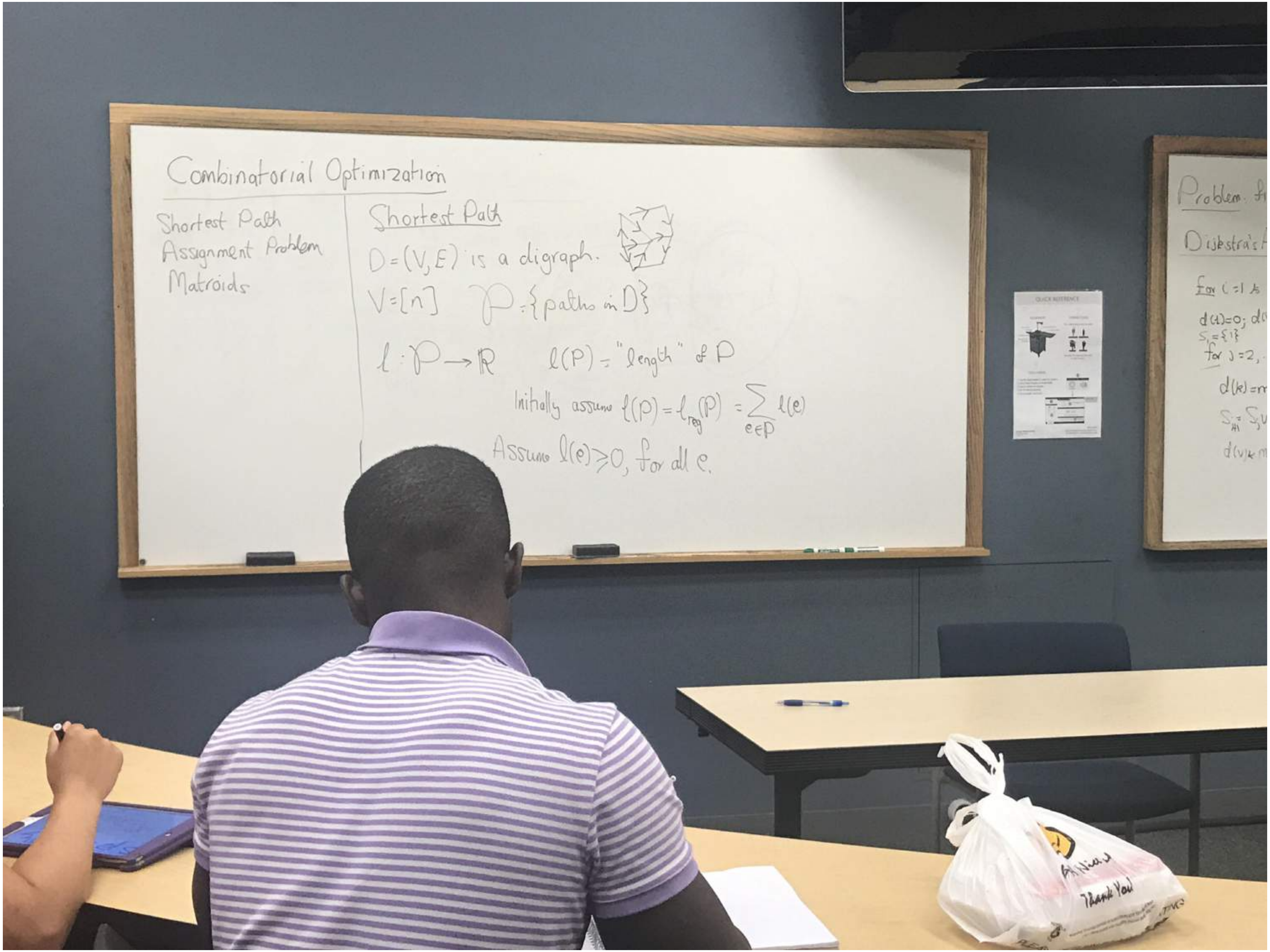
(1) If $(*)$ holds and $\hat{\pi}$ is any other policy with values $\hat{y}_i$ then $y_i \le \hat{y}_i$ for all i.

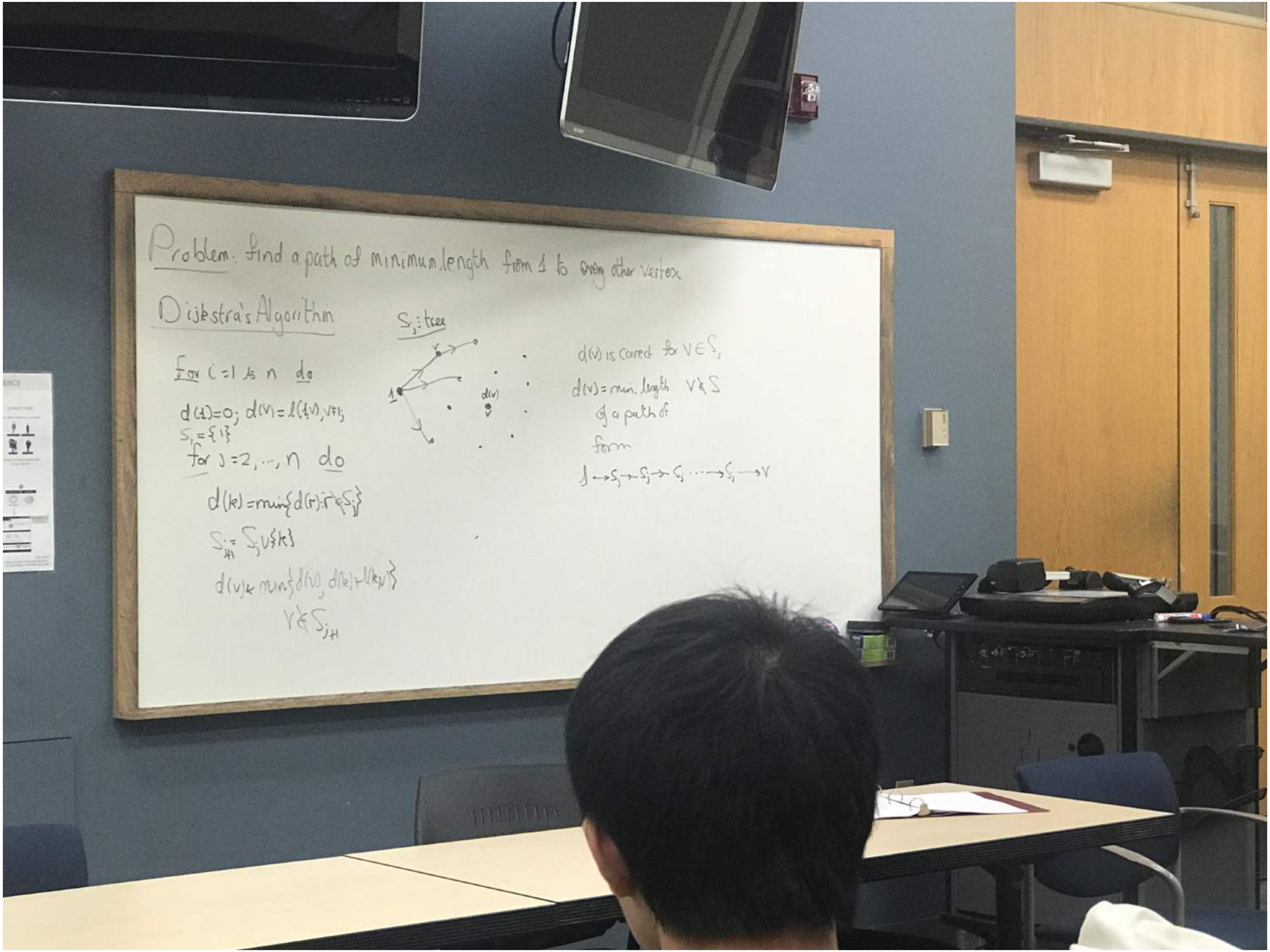
$$\hat{y}_i = c(i, \hat{\pi}(i)) + \alpha y_{\hat{\pi}(i)}$$

$$(*) \Rightarrow y_i \le c(i, \hat{\pi}(i)) + \alpha y_{\hat{\pi}(i)}$$

$$\xi_i = y_i - \hat{y}_i \Rightarrow \xi_i \le \alpha \xi_{\hat{\pi}(i)} \le \alpha^2 \xi_{\hat{\pi}(i)} \le \cdots \to 0$$


(11) If (*) does not hold then we can improve all y_i's:
Let $I = \{ i : y_i > c(i,\hat{\pi}(i)) + \alpha y_{\hat{\pi}(i)} = \min_j [c(i,j) + \alpha y_j]$
i=1 j c(i,j)+αy_j 1 8 * 2 10 3 8.5 4 9
i=2 j c(i,j)+αy_j 1 10 2 12 3 6.5 * 4 11
i=3 j c(i,j)+αy_j 1 12 2 11 3 8.5 * 4 9
i=4 j c(i,j)+αy_j 1 11 2 10 3 9.5 * 4 14
I={1,2,3,4}
New policy
$i \to \hat{\pi}(i)$ $i \in I$
$i \to \pi(i)$ $i \notin I$


Combinatorial Optimization
Shortest Path
Assignment Problem
Matroids
Shortest Path
D=(V,E) is a digraph.
V=[n]
P = {paths in D}
l : P → R
l(P) = "length" of P
Initially assume l(P) = l_reg(P) = Σ_{e∈P} l(e)
Assume l(e) ≥ 0, for all e.
Problem
Dijkstra's
Thank You

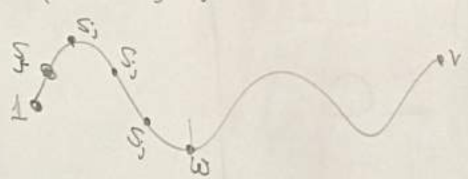

Problem: find a path of minimum length from 1 to every other vertex
Dijkstra's Algorithm
For i = 1 to n do
d(1)=0; d(v) = l(1,v), v≠1;
S1 = {1}
for j = 2, ..., n do
d(k) = min{d(r): r∉Sj}
Sj+1 = Sj ∪ {k}
d(v) ← min{d(v), d(k)+l(k,v)}
v∉Sj+1
Sj: tree
1
v
d(v)
d(v) is correct for v ∈ Sj
d(v) = min. length v∉S
of a path of
form
1 → Sj → Sj → Sj ··· → Sj → v

Claim. On termination, $d(v) =$ length of shortest path.

Suppose P is any other path from s to v.

$P = \{x_0 = 1, x_1, x_2, \ldots, x_r = v\}$

$$l(P) \geq l(P(1,w)) \geq d(w) \geq d(v).$$



Suppose v is added at step j