Matroids and the Greedy Algorithm

Independence System:

Ground set E.

$\mathcal{I} \subseteq 2^E$.

$J \subseteq I \in \mathcal{I} \Rightarrow J \in \mathcal{I}$

(i) Matchings

(ii) Forest Matroid

(iii) Stable Sets

(iv) 0-1 Knapsack Problem

(v) Matrices Matroid

maximise $\sum p_i x_i$ p_i means the weight

s.t. $\sum w_i x_i \le W$

$x_i = 0$ or 1

Greedy

$p_1 \geq p_2 \geq \cdots \geq p_n$

Try and add elements one at a time.

Optimization Problem

Given $E, \mathcal{I}$ and $w : E \to \mathbb{R}^+$

Problem: find $I \in \mathcal{I}$ that maximises $w(I) = \sum_{e \in I} w(e)$.

Greedy Algorithm [Myopic]

$E = \{e_1, e_2, \cdots, e_m\}$ and assume that $w(e_i) \geq w(e_{i+1}), \forall i$

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S = ∅
for i = 1 to m do
begin
  if S ∪ {e_i} ∈ I then S ← S ∪ {e_i}
end
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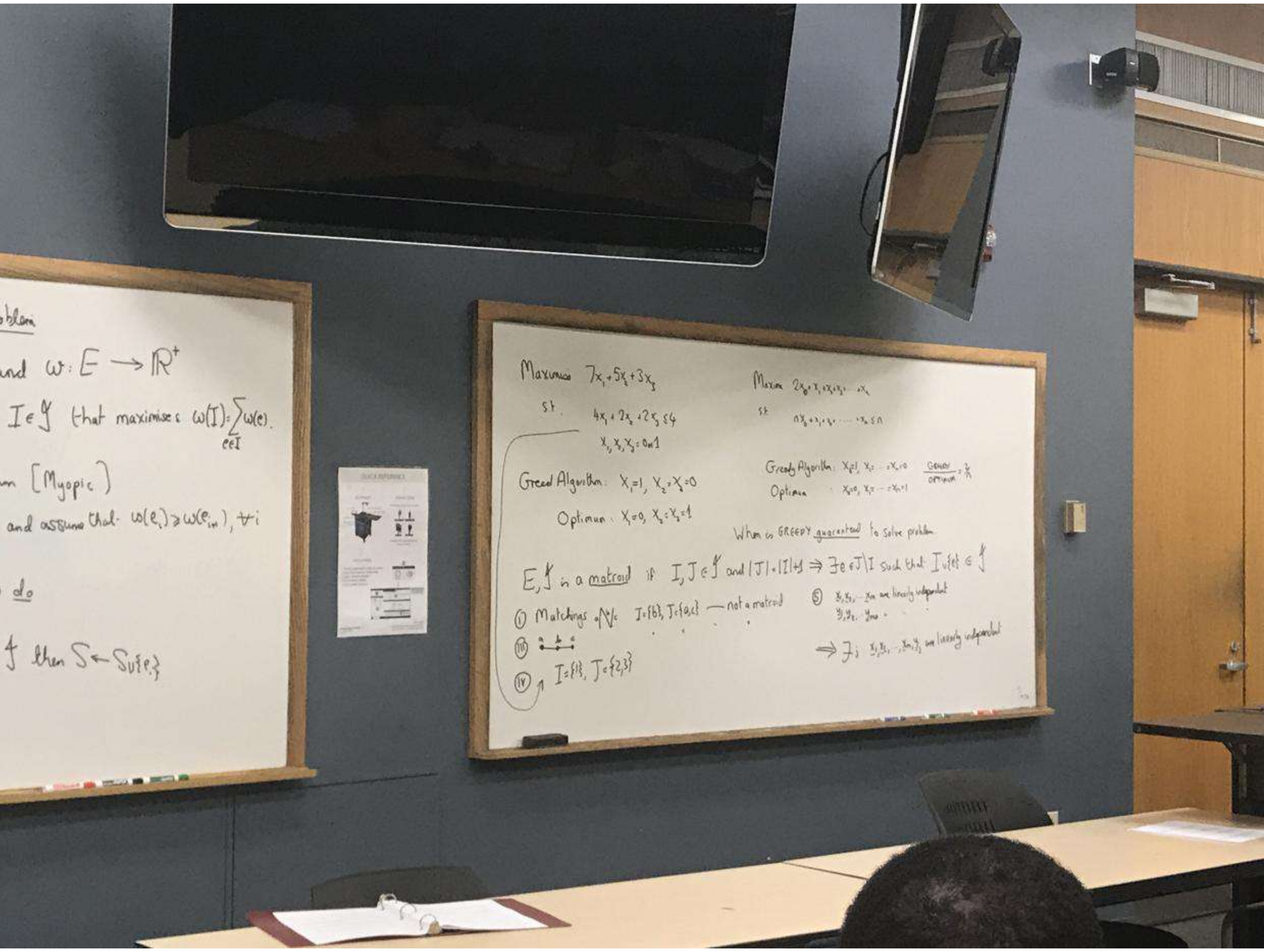
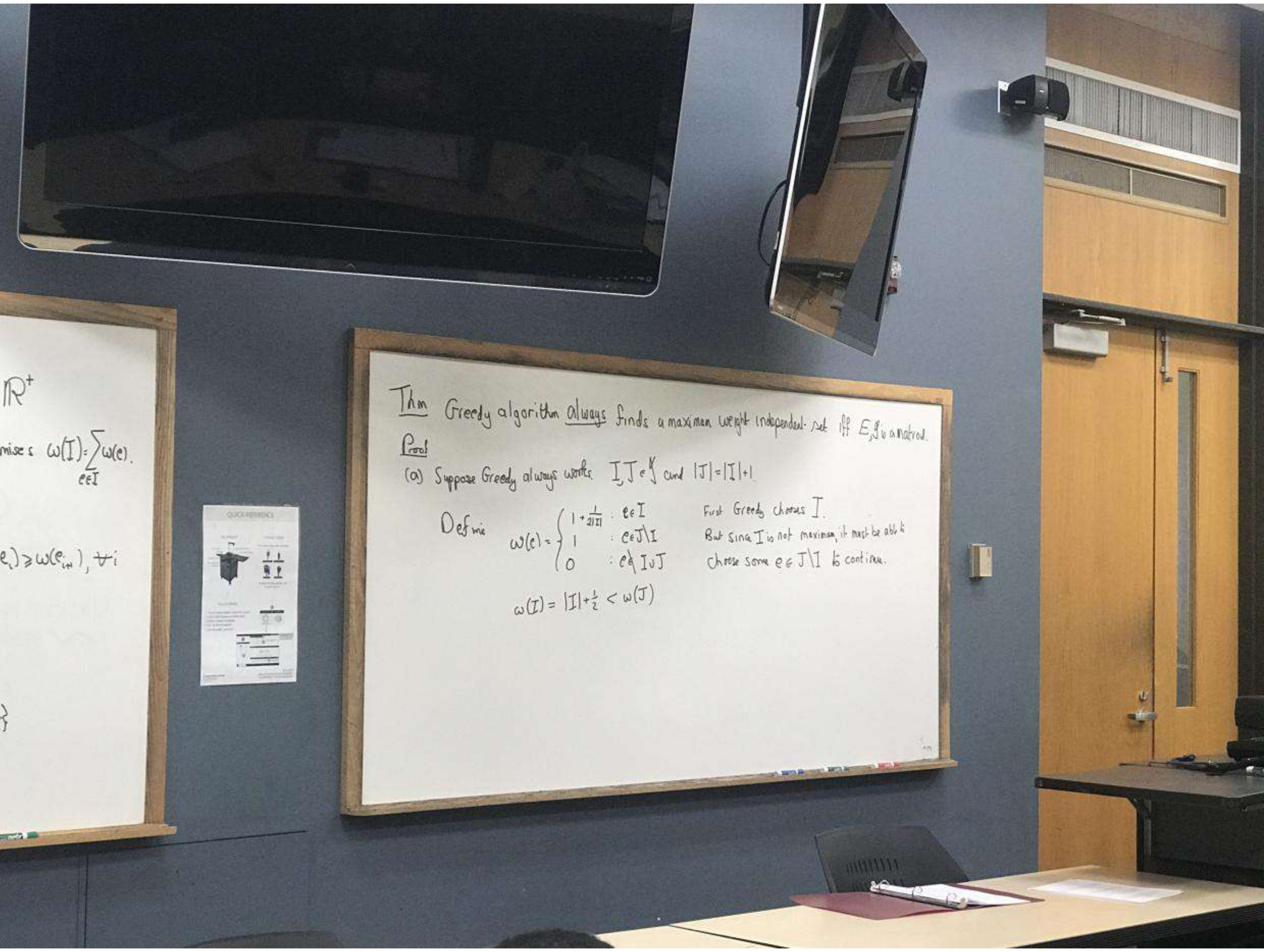


blem
and $w: E \to \mathbb{R}^+$
$I \in \mathcal{J}$ that maximises $w(I) = \sum_{e \in I} w(e)$.
[Myopic]
and assume that $w(e_i) \geq w(e_{i+1})$, $\forall i$
do
then $S \leftarrow S \cup \{e_i\}$
Maximise $7x_1 + 5x_2 + 3x_3$
s.t. $4x_1 + 2x_2 + 2x_3 \leq 4$
$x_1, x_2, x_3 = 0$ or 1
Greed Algorithm: $x_1 = 1$, $x_2 = x_3 = 0$
Optimum: $x_1 = 0$, $x_2 = x_3 = 1$
When is GREEDY guaranteed to solve problem
$E, \mathcal{J}$ is a matroid if $I, J \in \mathcal{J}$ and $|J| = |I| + 1 \Rightarrow \exists e \in J \setminus I$ such that $I \cup \{e\} \in \mathcal{J}$
Matchings
$I = \{b\}$, $J = \{a, c\}$ — not a matroid
$I = \{1\}$, $J = \{2, 3\}$
linearly independent



$\mathbb{R}^+$
mises $\omega(I) = \sum_{e \in I} \omega(e)$.
$e_i) \geq \omega(e_{i+1}), \forall i$
Thm Greedy algorithm always finds a maximum weight independent set iff $E, \mathcal{I}$ is a matroid.
Proof
(a) Suppose Greedy always works. $I, J \in \mathcal{I}$ and $|J| = |I| + 1$
Define $\omega(e) = \begin{cases} 1 + \frac{1}{2|I|} & : e \in I \\ 1 & : e \in J \setminus I \\ 0 & : e \notin I \cup J \end{cases}$
First Greedy chooses I.
But since I is not maximum, it must be able to
choose some $e \in J \setminus I$ to continue.
$\omega(I) = |I| + \frac{1}{2} < \omega(J)$

(b) Suppose $E, \mathcal{I}$ is a matroid.

Assume $\omega(e) > 0, \forall e \in E$

To maximise $\omega(I)$, we do not need to use elements e, with $\omega(e) \leq 0$

Also removing elements of weight ≤ 0, leaves a matroid.

Suppose Greedy chooses $I_G = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$

and $I = \{e_{j_1}, e_{j_2}, \ldots e_{j_\ell}\}$ is any other independent set

Optimization Problem

Given $E, \mathcal{I}$ and $\omega : E \to \mathbb{R}^+$

Problem: find $I \in \mathcal{I}$ that maximises $\omega(I) = \sum_{e \in I} \omega(e)$.

Greedy Algorithm [Myopic]

$E = \{e_1, e_2, \ldots, e_m\}$ and assume that $\omega(e_i) \geq \omega(e_{i+1}), \forall i$

$S = \emptyset$

for $i = 1$ **to** m **do**

begin

if $S \cup \{e_i\} \in \mathcal{I}$ then $S \leftarrow S \cup \{e_i\}$

end

We can assume $k=l$.

We show that $i_1 \le j_1$, $i_2 \le j_2$, ... $i_k \le j_k$

Suppose not and let t be the first index such that $i_t > j_t$

$$I = \{e_{i_1}, e_{i_2}, \dots, e_{i_{t-1}}\} \quad \text{and} \quad J = \{e_{j_1}, e_{j_2}, \dots, e_{j_t}\}$$

Imagine Greedy after $t-1$ rounds. Greedy chooses i_t

but there exist $e_{j_s} \in J \setminus I$ that could be added $j_s \le j_t < i_t$??