

ALP

$$\text{Minimize } \sum_i \sum_j c_{ij} x_{ij}$$

$$\text{s.t. } \sum_{j \in B} x_{ij} = 1, \forall i$$

$$\sum_{i \in A} x_{ij} = 1, \forall j$$

$$x_{ij} \geq 0$$

QLP

$$\text{Maximize } \sum_{i \in A} u_i + \sum_{j \in B} v_j$$

$$\text{s.t. } u_i + v_j \leq c_{ij}$$

Complementary Slackness

$$x_{ij} > 0 \Rightarrow u_i + v_j = c_{ij}$$

↓
 x optimal
 u, v optimal

① Choose an initial dual feasible u, v [$u_i = 0, \forall i \in A, v_j = \min_{i \in A} c_{ij}, \forall j \in B$]

② $K_{u,v}$ = bipartite graph with an edge (i,j) whenever $u_i + v_j = c_{ij}$

③ Find a maximum matching M in $K_{u,v}$

④ If M is perfect we are done.

⑤ If M is not perfect, change u, v and go to ②

$$\theta = \min \{ c_{ij} - u_i - v_j : (i \in A_M, j \notin B_M) \} > 0$$

$$u_i^* = \begin{cases} u_i + \theta & i \in A_M \\ u_i & i \notin A_M \end{cases} \quad v_j^* = \begin{cases} v_j - \theta & j \in B_M \\ v_j & j \notin B \end{cases}$$

(iii)

(a) Let M be a matching of $K_{A,B}$ [for example $M = \emptyset$]

(b) Orient edges of $K_{A,B}$:

Edges of $M: B \rightarrow A$

Edges of $M^c: A \rightarrow B$



All paths are alternating.
Want path from A_U to vertex of B not covered by M .

(c) A_U = vertices of A not covered by M .

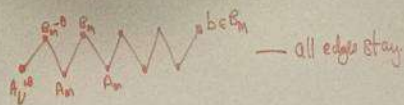
(d) A_M, B_M = set of vertices in A, B that are reachable by a path from A_U — e.g. BFS

(e) If there is $b \in B_M$ not covered by M then we have found an augmenting path. Increase size of M to b . Otherwise (iii) is over.



1. U^x, V^x is feasible. By our choices of θ

2. If $b \in B_M$, it stays in B_M after change to U^x, V^x



3. B_M grows by at least one.

So after doing ① at most n times, size of matching increases.



② The edge sets of the forests of a graph (V, X)

③ The set of stable sets of a graph (X, E)

$S \subseteq X$ is stable if it contains no edges

④ The solutions to the 0-1 knapsack problem.

Given w_1, w_2, \dots, w_n, W positive real numbers

$$J = \left\{ S \subseteq [n] : \sum_{i \in S} w_i \leq W \right\}$$

⑤ a_1, a_2, \dots, a_n are the columns of an $m \times n$ matrix A .

$X = [n]$, $J = \{ S : a_i \in S \text{ are linearly independent} \}$