ALP

$$\text{Minimise} \sum_{i} \sum_{j} c_{ij} x_{ij}$$

s.t.

$$\sum_{j=1}^{n} x_{ij} = 1, \ \forall i$$

$$\sum_{i=1}^{n} x_{ij} = 1, \ \forall j$$

$$x_{ij} \geq 0$$

DLP

$$\text{maximise} \sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$$

s.t. $u_i + v_j \leq c_{ij}$

Complementary Slackness

$$x_{ij} > 0 \Rightarrow u_i + v_j = c_{ij}$$

$$\Downarrow$$

$\underline{x}$ optimal

$\underline{u}, \underline{v}$ optimal

(I) Choose an initial dual feasible $\underline{u}, \underline{v}$ $[\, u_i = 0, \ i = 1, 2, \ldots, n \quad v_j = \min c_{ij}, \ j = 1, 2, \ldots, n \,]$

(II) $K_{\underline{u}, \underline{v}}$ = bipartite graph with an edge (i, j) whenever $u_i + v_j = c_{ij}$

(III) Find a maximum matching M in $K_{\underline{u}, \underline{v}}$

(IV) If M is perfect we are done.

(V) If M is not perfect, change $\underline{u}, \underline{v}$ and go to (III)

$$\theta = \min \{ c_{ij} - u_i - v_j : i \in A_M, j \notin B_M \} > 0$$

$$u_i^* = \begin{cases} u_i + \theta : i \in A_M \\ u_i : i \notin A_M \end{cases} \qquad v_j^* = \begin{cases} v_j - \theta : j \in B_M \\ v_j : j \notin B \end{cases}$$

III

(a) Let M be a matching of $K_{U,V}$ [For example $M = \emptyset$]

(b) Orient edges of $K_{U,V}$:

Edges of M: $B \to A$

Edges of M^c: $A \to B$

(c) A_U = vertices of A not covered by M.

(d) A_M, B_M = Set of vertices in A, B that are reachable by a path from A_U — e.g. BFS

(e) If there is $b \in B_M$ not covered by M then we have found an augmenting path. Increase size of M. Go to (b). Otherwise III is over.

$K_{U,V}$

A_U

All paths are alternating.

Want a path from A_U to a vertex of B not covered by M.

1. $\underline{u}^*, \underline{v}^*$ is feasible. By our choices of θ

2. If $b \in B_M$, it stays in B_M after change to $\underline{u}^*, \underline{v}^*$

— all edges stay.

3. B_M grows by at least one.

So after doing (V) at most n times, size of matching increases.

(2) The edge sets of the forests of a graph (V, X)

(3) The set of stable sets of a graph (X, E)

$S \subseteq X$ is stable if it contains no edges

(4) The solutions to the 0-1 Knapsack problem.

Given $w_1, w_2, \dots, w_n, W$ positive real numbers

$$\mathcal{J} = \left\{ S \subseteq [n] : \sum_{i \in S} w_i \le W \right\}$$

(5) $\underline{c}_1, \underline{c}_2, \dots, \underline{c}_n$ are the columns of an $m \times n$ matrix A.

$X = [n]$, $\mathcal{J} = \{ S : \underline{c}_i, i \in S$ are linearly independent$\}$