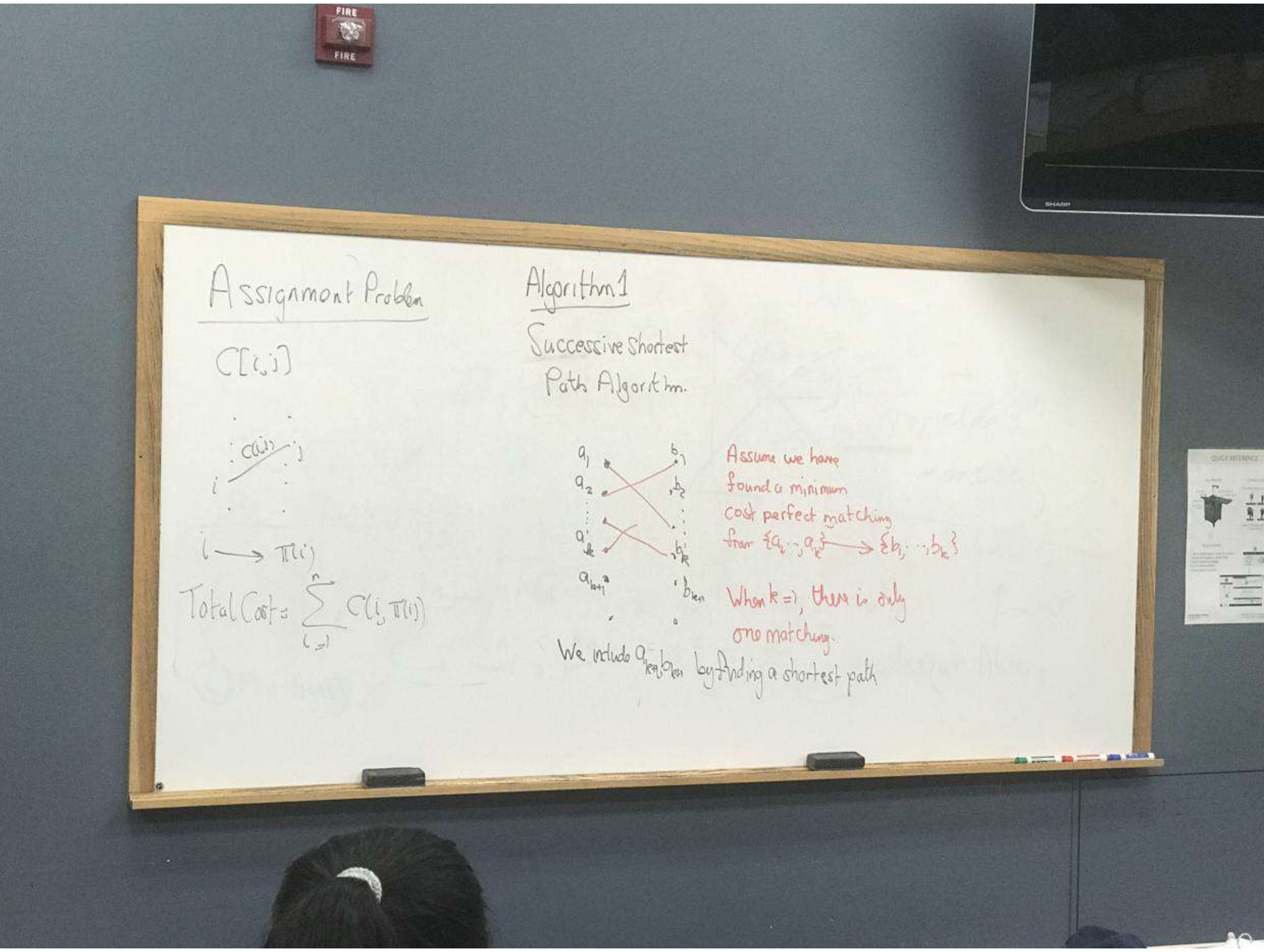
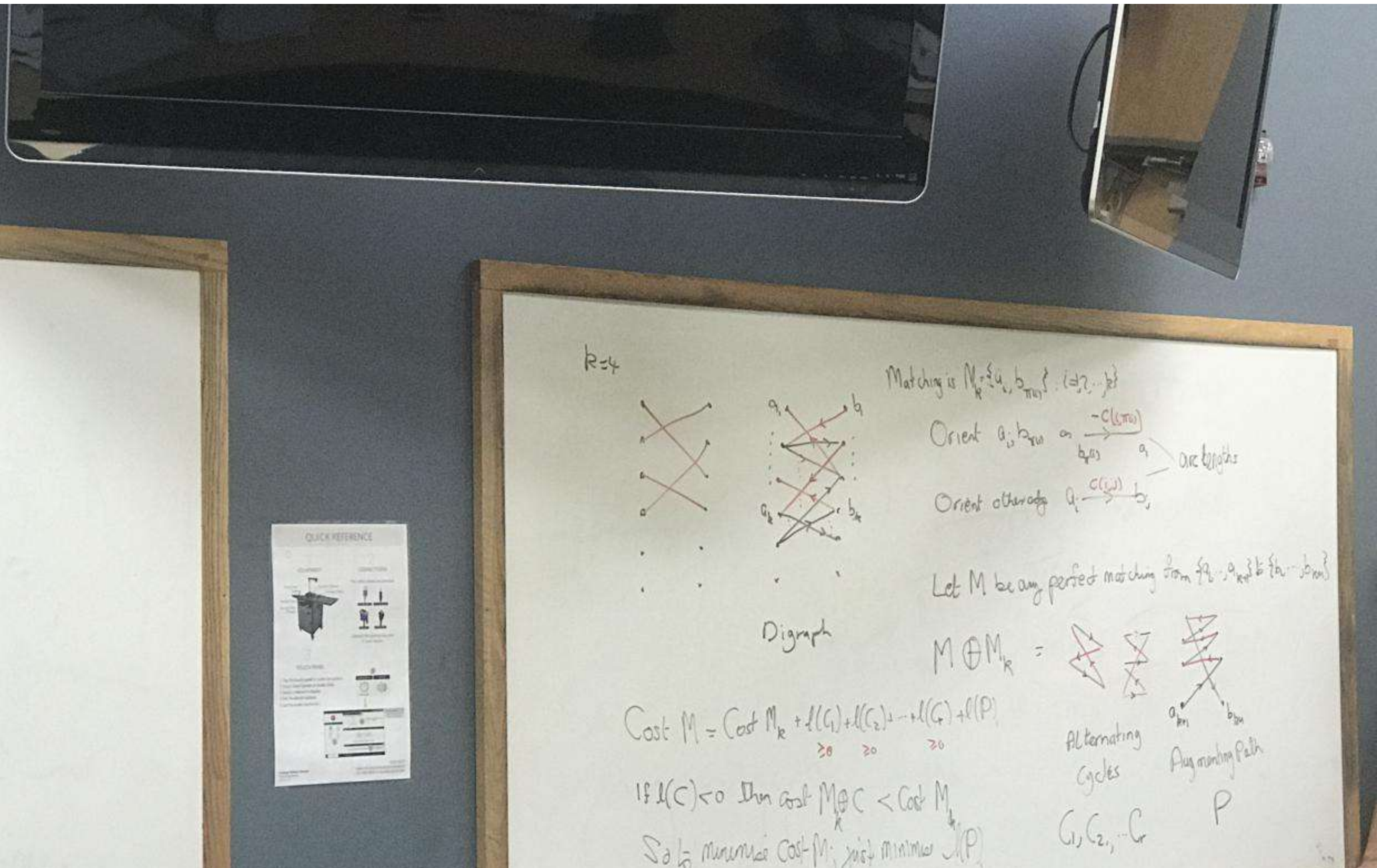
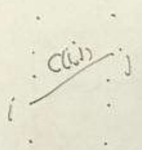

FIRE
FIRE
Assignment Problem
C[i,j]
c(i,j)
i
j
i → π(i)
Total Cost = Σ_{i=1}^{n} C(i, π(i))
Algorithm 1
Successive Shortest Path Algorithm.
a_1
a_2
a_k
a_{k+1}
b_1
b_2
b_k
b_{k+1}
Assume we have found a minimum cost perfect matching from {a_1, …, a_k} → {b_1, …, b_k}
When k = 1, there is only one matching.
We include a_{k+1}, b_{k+1} by finding a shortest path



$k=4$
Matching is $M_k = \{a_i, b_{\pi(i)}\} : i=1,2,\dots,k\}$
Orient $a_i, b_{\pi(i)}$ as $b_{\pi(i)} \xrightarrow{-c(i,\pi(i))} a_i$
arc lengths
Orient other edges $a_i \xrightarrow{c(i,j)} b_j$
Digraph
Let M be any perfect matching from $\{a_1, \dots, a_{k+1}\}$ to $\{b_1, \dots, b_{k+1}\}$
$M \oplus M_k =$
a_{k+1}
b_{k+1}
Cost M = Cost $M_k + \ell(C_1) + \ell(C_2) + \dots + \ell(C_r) + \ell(P)$
≥ 0 ≥ 0 ≥ 0
Alternating Cycles
$C_1, C_2, \dots C_r$
Augmenting Path
P
If $\ell(C) < 0$ then cost $M_k \oplus C$ < Cost M_k
So to minimise Cost M_j just minimise $\ell(P)$

Assignment Problem

$C[i,j]$



$i \longrightarrow \pi(i)$

Total Cost $= \sum_{i=1}^{n} C(i, \pi(i))$

Algorithms

Hungarian Algorithm or Primal Dual Algorithm

ALP: Minimize $\sum_{i=1}^{n} \sum_{j=1}^{n} C(i,j) x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \dots, n$

$\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2, \dots, n.$

$x_{ij} \geq 0$

Assignment $x_{ij} = 0$ or 1 $\quad$ $x_{ij} = 1$ iff $\pi(i) = j$

Integer Program

As it turns out, we can solve ALP and have a solution with $x_{ij} = 0$ or 1

DLP

Maximise $\sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$

Subject to $u_i + v_j \le c_{ij}$, for all i,j

Complementary Slackness

If a feasible solution $\underline{X}$ to ALP and a feasible solution $\underline{u},\underline{v}$ to DLP satisfy

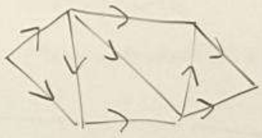
$$x_{ij} > 0 \text{ implies } u_i + v_j = c_{ij}$$

then $\underline{X}$ is optimal for ALP and $\underline{u},\underline{v}$ is optimal for DLP

$$0 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[c(i,j) - u_i - v_i\right] x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} c(i,j)\, x_{ij} - \sum_{i=1}^{n} u_i \underbrace{\sum_{j=1}^{n} x_{ij}}_{1} - \sum_{j=1}^{n} v_j \underbrace{\sum_{i=1}^{n} x_{ij}}_{1}$$

Acyclic Digraphs (DAG)

A DAG is a digraph with no directed circuits.



Topological Ordering

$D=(V,E)$ is a digraph

$V=\{\underbrace{v_1, v_2, \ldots, v_n}_{\text{topological ordering}}\}$ if $(v_i, v_j) \in E \Rightarrow i<j$

Theorem

A digraph D has a topological ordering iff it is a DAG.

Proof

Suppose $v_1, v_2, \ldots, v_n$ is a top. ord.

Suppose $C=(v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ is a directed cycle

$\Rightarrow i_1 < i_2 < i_3 < \ldots < i_k < i_1$ Contradiction