Optimization

$S = \emptyset$: unconstrained
$S \neq \emptyset$: constrained

Minimize $f(\underline{x})$

subject to $\underline{x} \in S \subseteq \mathbb{R}^n$

Linear Programming: $f(\underline{x}) = \underline{c}^T \underline{x}$ and $S = \{ A\underline{x} = \underline{b}, x \geq 0 \}$

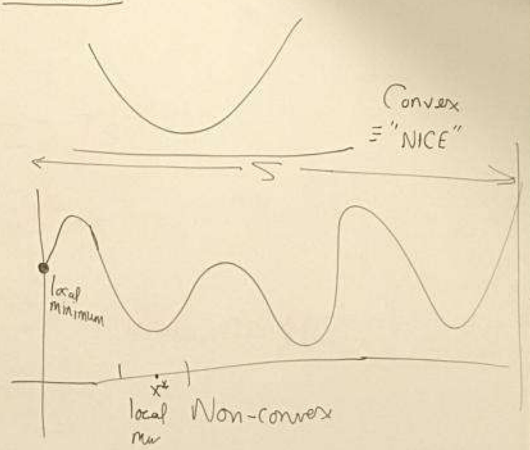
$\underline{x}^* \in S$ is a global minimum if $f(\underline{x}) \geq f(\underline{x}^*)$, for all $\underline{x} \in S$.

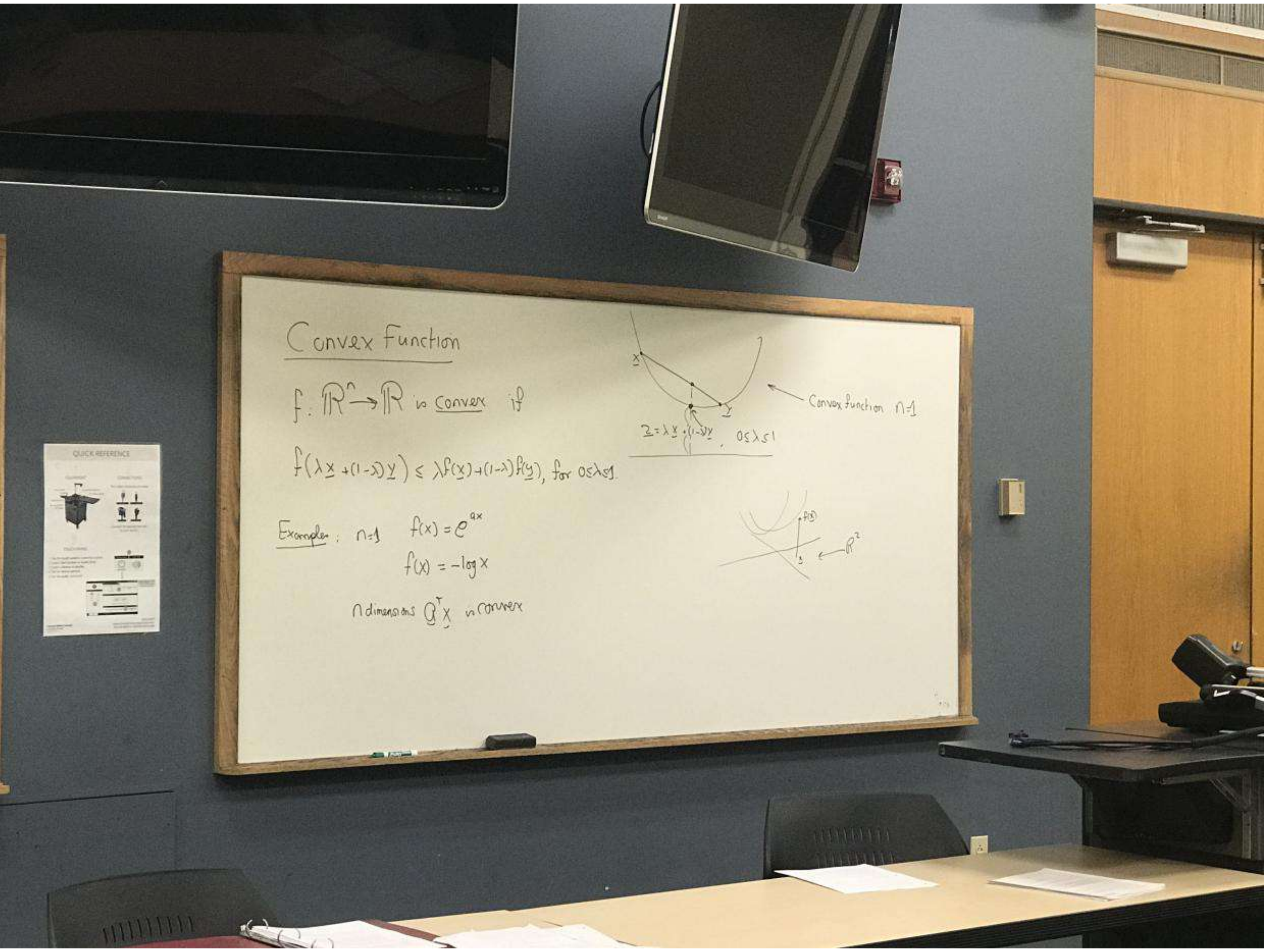
$\underline{x}^*$ is a local minimum if $f(\underline{x}^*) \leq f(\underline{x})$ for all $\underline{x}$ in a neighborhood of $\underline{x}^*$

i.e. $\exists \delta > 0$ such that $f(\underline{x}^*) \leq f(\underline{x})$ for all $\underline{x} \in B(\underline{x}^*, \delta) \cap S$

$B(\underline{x}^*, \delta) = \{ y : |\underline{y} - \underline{x}^*| \leq \delta \}$

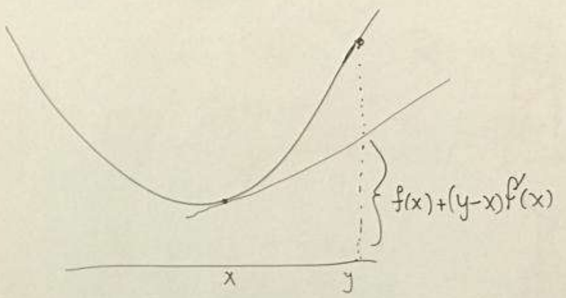
Minimize




Convex Function
f: ℝⁿ → ℝ is convex if
f(λx + (1−λ)y) ≤ λf(x) + (1−λ)f(y), for 0≤λ≤1.
Examples: n=1 f(x) = e^{ax}
f(x) = −log x
n dimensions aᵀx is convex
x
y
z = λx + (1−λ)y, 0≤λ≤1
Convex function n=1
f(z)
z
R²
QUICK REFERENCE

$n=1$

f is convex iff $f(y) \geq f(x) + (y-x)f'(x), \quad \forall x,y$



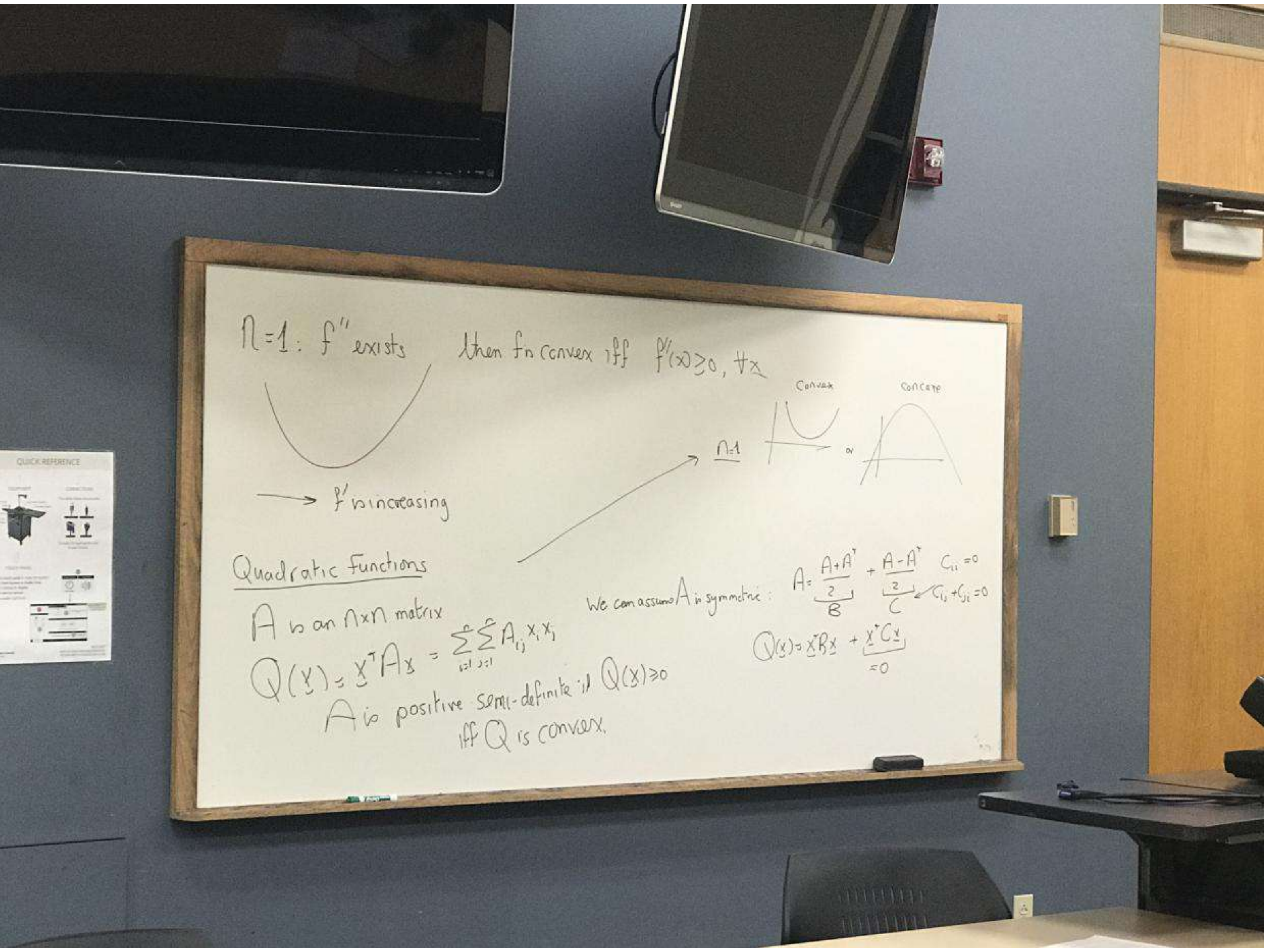
$\Longrightarrow$

$n \geq 1$

$$f(\underline{y}) \geq f(\underline{x}) + \nabla f(\underline{x}) \cdot (\underline{y} - \underline{x})$$

Consider $h(t) = f(t\underline{x} + (1-t)\underline{y})$

Convex function of t.

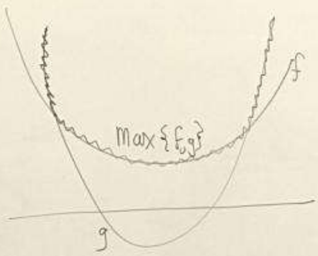

n=1 : f'' exists then fn convex iff f''(x) ≥ 0, ∀x
f' is increasing
Convex
Concave
n=1
or
Quadratic functions
A is an n×n matrix
Q(x) = xᵀAx = Σᵢ₌₁ Σⱼ₌₁ Aᵢⱼ xᵢ xⱼ
A is positive semi-definite if Q(x) ≥ 0
iff Q is convex.
We can assume A is symmetric : A = (A+Aᵀ)/2 + (A−Aᵀ)/2
B
C
Cᵢᵢ = 0
Cᵢⱼ + Cⱼᵢ = 0
Q(x) = xᵀBx + xᵀCx
=0

Operations on convex functions

(I) f, g convex $\Rightarrow f + g$ is convex

(II) f convex, $\lambda > 0 \Rightarrow \lambda f$ is convex

(III) f, g convex $\Rightarrow \max\{f, g\}$ is convex



Jensen's Inequality

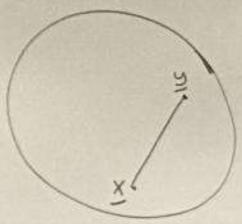
$\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$
$\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$

$f(\lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2 + \cdots + \lambda_m \underline{a}_n) \leq \lambda_1 f(\underline{a}_1) + \cdots + \lambda_n f(\underline{a}_n)$ $\leftarrow$ Induction on m.

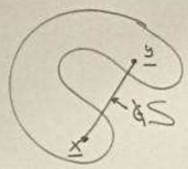
f is convex

$m = 2$, base case

Convex Sets



Non-convex



Examples

$S = \{\underline{x} : \underline{a}^T\underline{x} = 1\}$

$S = \{\underline{x} : \underline{a}^T\underline{x} \leq \underline{1}\}$

$S = B(\underline{x}, \delta)$

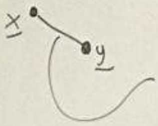
Level Sets of f.

$\{\underline{x} : f(\underline{x}) \leq 0\}$ where f is convex

$$f(\lambda\underline{x} + (1-\lambda)\underline{y}) \leq \lambda \underbrace{f(\underline{x})}_{\leq 0} + (1-\lambda)\underbrace{f(\underline{y})}_{\leq 0}$$

≤ 0

Convex: S is Convex iff for line segment $L(\underline{x}, \underline{y})$, $\underline{x}, \underline{y} \in S \Rightarrow L(\underline{x}, \underline{y}) \subseteq S$.



$\{\lambda\underline{x} + (1-\lambda)\underline{y} : 0 \leq \lambda \leq 1\}$

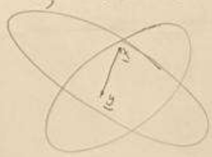
Operations on Convex Sets

(i) 

$\longrightarrow \underline{y}+S = \{\underline{y}+\underline{x} : \underline{x}\in S\}$

Convex

(ii) S, T are convex then $S\cap T$ is convex



If f, S are convex and $\underline{x}^*$ is a local minimum of f then $\underline{x}^*$ is also a global minimum.