Lagrangean Duality

Problem: minimize $f(\underline{x})$ $\quad \underline{x} \in \mathbb{R}^n$

subject to $g_i(\underline{x}) \leq 0 \quad i = 1, 2, \cdots, m$

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \sum_{i=1}^{m} \lambda_i g_i(\underline{x})$$

Lagrangean

$$\phi(\underline{\lambda}) = \min_{\underline{x} \in \mathbb{R}^n} L(\underline{x}, \underline{\lambda})$$

ϕ is a concave function

Dual Problem

maximize $\Phi(\underline{\lambda}) : \underline{\lambda} \geq 0$

Ex: Linear Programming

$$f(\underline{x}) = \underline{c}^T \underline{x} \ \& \ g_i(\underline{x}) = -\underline{a}_i^T \underline{x} + b_i$$

$$L(\underline{x}, \underline{\lambda}) = \underline{c}^T \underline{x} + \sum_{i=1}^{m} \lambda_i \left[-\underline{a}_i^T \underline{x} + b_i \right]$$

$$= \left[\underline{c}^T - \underline{\lambda}^T A \right] \underline{x} + \underline{\lambda}^T \underline{b}$$

$$\phi(\underline{\lambda}) = \begin{cases} \underline{\lambda}^T \underline{b} & \text{if } \underline{\lambda}^T A = \underline{c}^T \\ -\infty & \text{if } \underline{\lambda}^T A \neq \underline{c}^T \end{cases}$$

LP: min $\underline{c}^T \underline{x}$

s.t. $A\underline{x} \geq \underline{b}$

$\underline{a}_i^T \underline{x} \geq b_i$

$-\underline{a}_i^T \underline{x} + b_i \leq 0$

Dual Problem:

Maximize $\underline{b}^T \underline{\lambda}$

s.t. $A^T \underline{\lambda} = \underline{c}$

$\underline{\lambda} \geq 0$

Weak Duality

$$\phi(\underline{\lambda}) \le L(\underline{x}, \underline{\lambda}) \le f(\underline{x}).$$

$\forall \underline{x}$

$\underline{x}$ is feasible for primal & $\underline{\lambda} \ge 0$

Strong Duality

Sometimes $\max_{\underline{\lambda} \ge 0} \phi(\underline{\lambda}) = \min_{\underline{x} : \underline{g}(\underline{x}) \le 0} f(\underline{x})$

Notes: if $\exists\, \underline{x}$ s.t. $g_i(\underline{x}) < 0$, $i = 1, 2, \dots, m$

Conditions for local optimality - necessary

Unconstrained case: $\nabla f(\underline{x}) = 0$

Otherwise $f(\underline{x} + \underline{h}) = f(\underline{x}) + \nabla f(\underline{x}) \cdot \underline{h} + O(\|h\|^2)$

if $\nabla f(\underline{x}) \ne 0$
and $\underline{h} = -t \nabla f(\underline{x})$, t small
then $f(\underline{x} + \underline{h}) < f(\underline{x})$.

Lagrangean Duality

Problem P: minimize $f(\underline{x})$ $\quad \underline{x} \in \mathbb{R}^n$

subject to $g_i(\underline{x}) \le 0 \quad i=1,2,\dots,m$

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \sum_{i=1}^{m} \lambda_i g_i(\underline{x})$$

"Necessary" conditions for local optimum in P.

Karusch-Kuhn-Tucker Conditions

$\underline{a}$ satisfies KKT

$$g_i(\underline{a}) \le 0 \quad i=1,2,\dots,m$$

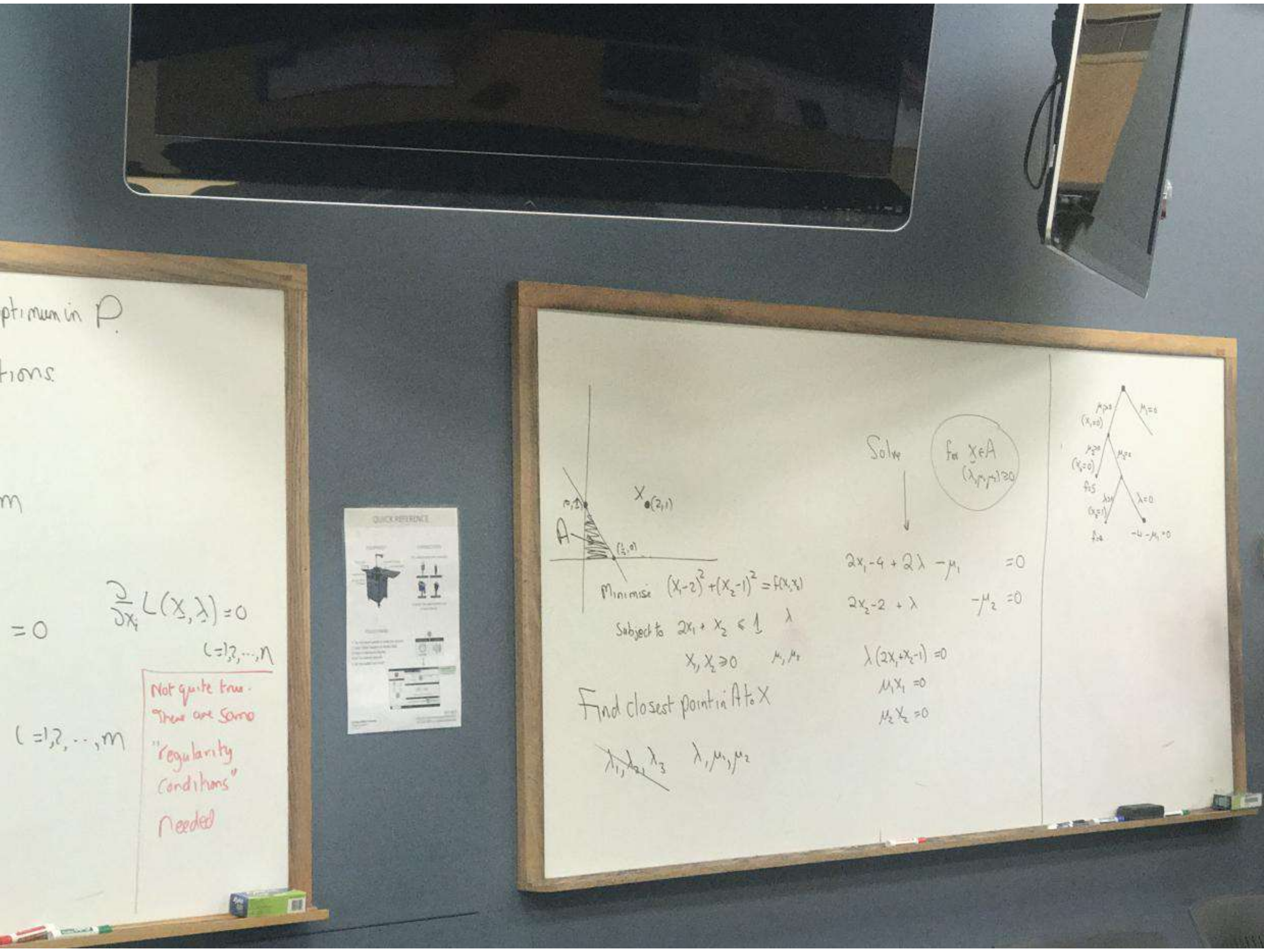
$$\underline{\lambda} \ge 0$$

$$\nabla f(\underline{a}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\underline{a}) = 0 \qquad \frac{\partial}{\partial x_i} L(\underline{x}, \underline{\lambda}) = 0 \quad i=1,2,\dots,n$$

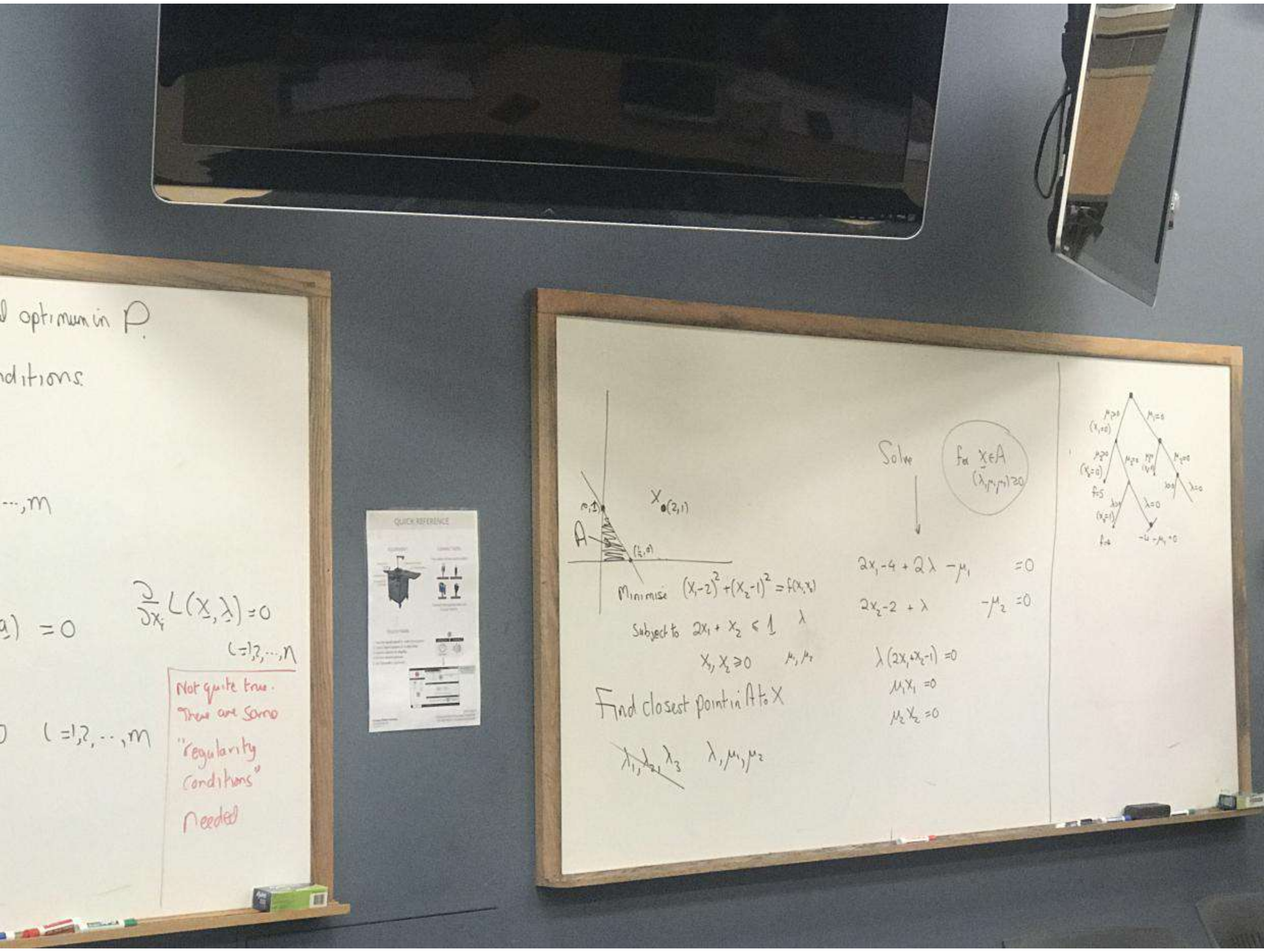
$$\lambda_i g_i(\underline{a}) = 0 \quad i=1,2,\dots,m$$

λ_i is positive only at tight constraints

Not quite true. There are some "regularity conditions" needed



ptimum in P.
tions
= 0
∂/∂x_i L(x, λ) = 0
i = 1,2,...,n
i = 1,2,...,m
Not quite true. There are some "regularity conditions" needed
QUICK REFERENCE
X (2,1)
A
Minimise (x1-2)^2 + (x2-1)^2 = f(x1,x2)
Subject to 2x1 + x2 ≤ 1 λ
x1, x2 ≥ 0 μ1, μ2
Find closest point in A to X
λ1, λ2, λ3 λ, μ1, μ2
Solve
for x∈A
(λ,μ1,μ2)≥0
2x1 - 4 + 2λ - μ1 = 0
2x2 - 2 + λ - μ2 = 0
λ(2x1+x2-1) = 0
μ1x1 = 0
μ2x2 = 0


optimum in P.
nditions.
..., m
$\frac{\partial}{\partial x_i} L(\underline{x}, \underline{\lambda}) = 0$
$i = 1, 2, \ldots, n$
$= 0$
$i = 1, 2, \ldots, m$
Not quite true.
There are some
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QUICK REFERENCE
$X \bullet (2,1)$
A
Minimise $(x_1-2)^2+(x_2-1)^2 = f(x_1,x_2)$
Subject to $2x_1 + x_2 \leq 1$ λ
$x_1, x_2 \geq 0$ μ_1, μ_2
Find closest point in A to X
$\cancel{\lambda_1, \lambda_2, \lambda_3}$ λ, μ_1, μ_2
Solve
for $\underline{x} \in A$
$(\lambda, \mu_1, \mu_2) \geq 0$
$2x_1 - 4 + 2\lambda - \mu_1 = 0$
$2x_2 - 2 + \lambda - \mu_2 = 0$
$\lambda(2x_1 + x_2 - 1) = 0$
$\mu_1 x_1 = 0$
$\mu_2 x_2 = 0$