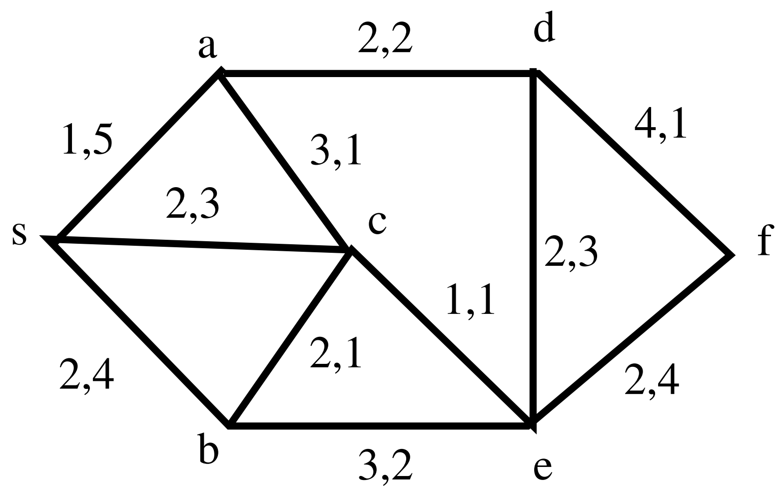
Department of Mathematical Sciences
**CARNEGIE MELLON UNIVERSITY**

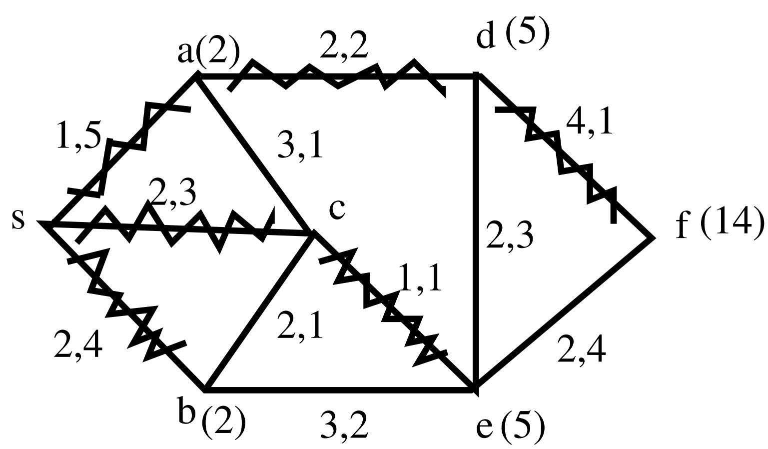
**OPERATIONS RESEARCH II 21-393**

Homework 2: Due Friday September 19.

1. Find a shortest path from $s$ to all other nodes in the digraph below. Each edge $(x, y)$ is labelled by a pair $(a, b)$ and the length of the corresponding arc is $a + bt$ where $t$ is the time the path reaches $x$. All arcs are directed lexicograhically e.g. $(c, e)$ is directed from $c$ to $e$.




**Solution**



2. Let $\mathcal{W}$ denote the set of walks in a directed graph $D$. If $W_1$ is a walk from $a$ to $b$ and $W_2$ is a walk from $b$ to $c$ then $W_1 + W_2$ is the walk from $a$ to $c$ obtained by following $W_1$ and then $W_2$.

   Let $\ell : \mathcal{W} \to \mathbb{R}$ be a real valued function defined on $\mathcal{W}$. Suppose that it has the following properties:

   (a) $\ell(C) \geq 0$ for any closed walk $C$. (A walk is closed if it begins and ends at the same vertex).

   (b) If $W_1, W_1'$ are walks from $a$ to $b$ and $W_2, W_2'$ are walks from $b$ to $c$ and $\ell(W_i') \geq \ell(W_i)$ for $i = 1, 2$ then $\ell(W_1' + W_2') \geq \ell(W_1 + W_2)$.

   Consider the following algorithm: $n$ is the number of vertices in $D$.
   **Initialise** $W_{i,j} = (i, j)$ and $D_{i,j} = \ell(W_{i,j})$ for $i, j = 1, 2, \ldots, n$.

   **For** $k = 1$ to $n$ **Do**
       **For** $i = 1$ to $n$ **Do**
           **For** $j = 1$ to $n$ **Do**
           $D_{i,j} \leftarrow \min\{D_{i,j}, \ell(W_{i,k} + W_{k,j})\}$
           **If** $D_{i,j} = \ell(W_{i,k} + W_{k,j})$ **then** $W_{i,j} \leftarrow W_{i,k} + W_{k,j}$
           **oD**
       **oD**
   **oD**

   Prove that when the algorithm finishes,

   $$D_{i,j} = \min\{\ell(P) : P \text{ is a path from } i \text{ to } j\}.$$

   **Solution:** We argue by induction on $k$ that at the end of $k$ executions of the outermost loop, for all $i, j$, $D_{i,j}$ minimises $\ell(P)$ over all walks from $i$ to $j$ whose interior vertices are in $\{1, 2, \ldots, k\}$.

   This is trivially true for $k = 0$, since the claim is about the "lengths" of the edges $(i, j)$.

   Assume that the claim is true for $k \geq 0$. A walk from $i$ to $j$ either uses vertex $k+1$ in its interior or it doesn't. By induction, the shortest walk from $i$ to $j$ that doesn't use $k + 1$ is $D_{i,j}$. So, we only have to argue that $\ell(W_{i,k+1} + W_{k+1,j})$ is the length of a shortest walk from $i$ to $k$ that uses $k + 1$.

Let $W = W_1 + W_2 + W_3$ be a walk from $i$ to $j$ where
$W_1$ goes from $i$ to $k+1$ and only uses $\{1, 2, \ldots, k\}$ in its interior;
$W_2$ goes from $k+1$ to $k+1$;
$W_3$ goes from $k+1$ to $j$ and only uses $\{1, 2, \ldots, k\}$ in its interior.

It follows from Property (b) that

$$\ell(W_1 + W_2) \geq \ell(W_1 + \Lambda) = \ell(W_1)$$

where $\Lambda$ is the path from $k+1$ to $k+1$ that consists of the single vertex $k+1$.

Applying (b) again, we see that

$$\ell(W) \geq \ell(W_1 + W_3) \geq \ell(W_{i,k+1} + W_{k+1,j}).$$

This completes the induction. So, for each $i, j$, $W_{i,j}$ minimises $\ell(W)$ over all walks from $i$ to $j$. Now if $W_{i,j}$ is not a path, then we can write it as $W_1 + W_2 + W_3$ as above and show that there is a walk $W'$ from $i$ to $j$ with $\ell(W') \leq \ell(W_{i,j})$ and with fewer edges. Clearly $\ell(W') = \ell(W_{i,j})$ here, but then we have that the walk from $i$ to $j$ that minimises $\ell$ and has fewest edges among walks that minimise $\ell$, must be a path.

3. Suppose that the edges of a connected graph $G = (V, E = \{e_1, e_2, \ldots, e_m\}$ are given lengths $\ell(e_i), i = 1, 2, \ldots, m$ where $\ell(e_i) < \ell(e_{i+1}), 1 \leq i < m - 1$. Fot two spanning trees $T_1, T_2$ we say that $T_1 \prec T_2$ if there exists $r \leq n - 1$ $(n = |V|)$ such that if $T_1 = \{e_{i_1}, e_{i_2}, \ldots, e_{i_{n-1}}\}$ and $T_2 = \{e_{j_1}, e_{j_2}, \ldots, e_{j_{n-1}}\}$ then $i_k = j_k$ for $1 \leq k < r$ and $i_r < j_r$.

   Prove that if $T^*$ is the tree constructed by the Greedy Algorithm (Kruskal) and $T$ is any other spanning tree, then $T^* \prec T$.

   **Solution:** This follows immediately from the fact that $e_{j_k}$ has minimum length of all edges that do not create cycles with $\{e_{j_1}, e_{j_2}, \ldots, e_{j_{k-1}}\}$