Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 4: Due Monday October 27.

1. Consider the following in relation to the 0-1 Knapsack problem: For $i \in [n]$ let M_i be the value of the solution where (i) we fix $x_i = 1$ and (ii) we apply the greedy algorithm to find the remaining items. Let $M = \max\{M_1, M_2, \ldots, M_n\}$. Show that $M \geq Z_{OPT}/2$.

   Solution: Let $x_1^*, x_2^*, \ldots, x_n^*$ denote an optimal solution. Assume that $c_1 = \max\{c_i : x_i^* = 1\}$ and that $c_i/w_i \geq c_{i+1}/w_{i+1}$ for $2 \leq i < n$.

   We now show that $M_1 \geq Z_{OPT}/2$. Let the solution associated with M_1 be $\widehat{x}_1 = 1, \widehat{x}_2, \ldots, \widehat{x}_n$ and let m be the first index i such that $\widehat{x}_i = 0$. Next let

$$
\begin{aligned}
A &= c_1 + \cdots + c_{m-1} \\
B &= W - (w_1 + \cdots + w_{m-1}) \\
C &= \sum_{i=2}^{m-1} w_i 1(x_i^* = 0).
\end{aligned}
$$

   Then we have

   (a) $M_1 \geq A$.

   (b) $B < w_m$, else $\widehat{x}_m = 1$.

   (c)

$$
\begin{aligned}
Z_{OPT} &\leq A - C\frac{c_{m-1}}{w_{m-1}} + \frac{c_m}{w_m}(B + C) \\
&< A + c_m + C\left(\frac{c_m}{w_m} - \frac{c_{m-1}}{w_{m-1}}\right) \\
&\leq A + c_m \\
&\leq A + c_1 \\
&\leq 2A.
\end{aligned}
$$

2. Give an algorithm to solve the following scheduling problem. There are n jobs labelled $1, 2, \ldots, n$ that have to be processed one at a time on a single machine. There is an acyclic digraph $D = (V, A)$ such that if $(i, j) \in A$ then job j cannot be started until job i has been completed. The problem is to minimise $\max_j f_j(C_j)$ where for all j, f_j is a monotone increasing. As usual, C_j is the completion time of job j. This is distinct from its processing time p_j.

   Solution: Because D is acyclic, we can assume that the $(i, j) \in A$ implies that $i < j$. Next let j be a sink if there are no edges in D of the form (j, k) i.e. directed from j to k.

   For a set of jobs S let $p(S) = \sum_{j \in S} p_j$.

   The algorithm is to process last, the sink j^* that minimises $f_j(p([n]))$. Then apply this procedure recursively to the remining jobs $[n] \setminus \{j^*\}$.

   Let $f^*(S)$ denote the optimum schedule value, if we only schedule jobs in S. Then we observe that

$$\begin{aligned} f^*([n]) &\geq \min_{j \in [n]} f_j(p([n])) \\ f^*([n]) &\geq f^*([n] \setminus \{j\}) \qquad \text{for all sinks } j \in [n]. \end{aligned}$$

   We use these inequalities to prove by induction that our schedule is optimal. According to our scheduling rule, we schedule last the job j minimizing $f_j(p([n]))$. By induction, this gives us a schedule with objective $\max\{f^{(}_j p([n])), f^*([n] \setminus \{j\})\}$. But since each of these quantities is (by the equations above) a lower bound on $f^*([n])$ we see that in fact we obtain a schedule whose value is a lower bound on $f^*([n])$, and thus must in fact equal $f^*([n])$.

3. Find the optimal ordering strategy for the following inventory system. If you order an amount Q, it costs AQ^α for some $0 < \alpha < 1$ and the inventory cost is I per unit per period. The demand is λ units per period and stock-outs are allowed. The penalty cost for stock-outs are π per unit per period.

   Solution: The total cost K is given by

$$K = \frac{\lambda A}{Q^{1-\alpha}} + \frac{I(Q - S)^2}{2Q} + \frac{\pi S^2}{2Q}.$$

We then have, at the minimum,

$$\frac{\partial K}{\partial S} = \frac{I(S-Q)}{Q} + \frac{\pi S}{Q} = 0$$

which implies that

$$S = \frac{IQ}{I+\pi}.$$

Then we have,

$$\begin{aligned}\frac{\partial K}{\partial Q} &= -\frac{lA(1-\alpha)}{Q^{2-\alpha}} + \frac{I(Q-S)}{Q} - \frac{I(Q-S)^2}{2Q^2} - \frac{\pi S^2}{2Q^2} \\ &= -\frac{lA(1-\alpha)}{Q^{2-\alpha}} + \frac{I\pi}{I+\pi} - \frac{I\pi^2}{2(I+\pi)^2} - \frac{\pi I^2}{2(I+\pi)^2} \\ &= -\frac{lA(1-\alpha)}{Q^{2-\alpha}} + \frac{I\pi}{2(I+\pi)} \\ &= 0,\end{aligned}$$

at the minimum. So, we have

$$\begin{aligned}Q &= \left(\frac{2lA(1-\alpha)(I+\pi)}{I\pi}\right)^{1/(2-\alpha)}. \\ S &= \frac{I}{I+\pi}\left(\frac{2lA(1-\alpha)(I+\pi)}{I\pi}\right)^{1/(2-\alpha)}. \\ K &= lA\left(\frac{I\pi}{2lA(1-\alpha)(I+\pi)}\right)^{(1-\alpha)/(2-\alpha)} + \frac{I\pi}{2(I+\pi)}\left(\frac{2lA(1-\alpha)(I+\pi)}{I\pi}\right)^{1/(2-\alpha)} \\ &= \left(\frac{2I\pi}{I+\pi}\right)^{(1-\alpha)/(2-\alpha)}(lA(1-\alpha))^{1/(2-\alpha)}.\end{aligned}$$