Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 3: Due Monday October 13.

1. A cloth manufacturer sells rolls of cloth in n widths $\ell_1, \ell_2, \ldots, \ell_n$. Production is only in widths of width L. The manufacturer has to meet demand for d_j rolls of width ℓ_j and these must be cut from the larger rolls. For example if $\ell_1 = 7$ and $\ell_2 = 5$ and $L = 36$ then the manufacturer can cut 4 rolls of width 7 and 1 roll of width 5 from one large roll, leaving 3 feet of waste.

   The manufacturer wishes to meet demand and minimise total waste. Write an Integer Programming Formulation for this problem. The manufacturer will have to cut up several rolls in several differnt ways to solve this problem.

   Solution: Let Z_+ denote the set of non-negative integers and let $X = \{x \in Z^n : \ \ell_1 x_1 + \cdots + \ell_n x_n \leq L\}$. If the manufacturer cuts up m_x rolls with pattern x then the total waste is

$$L \sum_{x \in X} m_x - \sum_{j=1}^{n} d_j \ell_j$$

   and so minimising $\sum_{x \in X} m_x$ is equivalent to minimising waste. The problem is therefore

$$\begin{array}{lll} \text{Minimise} & \sum_{x \in X} m_x & \\ \text{Subject to} & \ell_j \sum_{x \in X} m_x x_j \geq d_j & j = 1, 2, \ldots, n \\ & m_x \in Z_+ & \forall x \in X. \end{array}$$

2. Solve the following problem by a cutting plane algorithm:

$$\begin{array}{llllllll} \text{minimise} & 4x_1 & + & 5x_2 & +3x_3 & & \\ \text{subject to} & & & & & & \\ & 2x_1 & + & x_2 & -x_3 & \geq & 2 \\ & x_1 & + & 4x_2 & +x_3 & \geq & 13 \end{array}$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution

Initial tableau

x_1	x_2	x_3	x_4	x_5	R.H.S	
-4	-5	-3	0	0	0	z
-2	-1	1	1	0	-2	x_4
-1	-4	-1	0	1	-13	$x_5 \leftarrow$
	$\uparrow$					

x_1	x_2	x_3	x_4	x_5	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	$\frac{13}{4}$	x_2

Primal feasible, but the solution is not integral.

We add a cut which eliminates the current optimal solution.

$$\frac{1}{4}x_1 + \frac{1}{4}x_3 + \frac{3}{4}x_5 - y_1 = \frac{1}{4}$$

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	0	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	0	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	0	$\frac{13}{4}$	x_2
$\frac{-11}{4}$	0	$\frac{-1}{4}$	0	$\frac{-3}{4}$	+1	$\frac{-1}{4}$	$y_1 \leftarrow$
				$\uparrow$			

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	$\frac{-5}{3}$	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	$\frac{-1}{3}$	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{-1}{3}$	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	$\frac{-4}{3}$	$\frac{1}{3}$	x_5

The solution is primal feasible and so optimal but still not integer.

We add a cut which eliminates the current optimal solution.

$$\frac{-1}{3}x_1 - \frac{1}{3}x_3 + y_2 = \frac{1}{3}$$

We obtain tableau

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	0	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	0	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	x_5
$\frac{-1}{3}$	0	$\frac{-1}{3}$	0	0	1	$\frac{1}{3}$	$y_2 \leftarrow$
		$\uparrow$					

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
-1	0	0	0	0	-4	18	z
-3	0	0	1	0	4	0	x_4
0	1	0	0	0	1	3	x_2
0	0	0	0	1	1	0	x_5
1	0	1	0	0	-3	1	x_3

Which is optimal integral.

3. Solve the following problem by a branch and bound algorithm:

$$\begin{array}{llll} \text{Maximise} & x_1 & +2x_2 & \\ \text{subject to} & & & \\ & 2x_1 & +x_2 & \leq 7 \\ & -x_1 & +x_2 & \leq 3 \end{array}$$

$$x_1, x_2 \geq 0.$$

$$x_1, x_2 \text{ integer.}$$

Solution

1. LP relaxation:

$$(x_1, x_2) = \left(\frac{4}{3}, \frac{13}{3}\right) \qquad Value = 10.$$

Sub-problem 1: add constraint $x_1 \leq 1$.

$$(x_1, x_2) = (1, 4) \qquad Value = 9.$$

This problem is solved.

Sub-problem 2: add constraint $x_1 \geq 2$.

$$(x_1, x_2) = (2, 3) \qquad Value = 8.$$

This problem is solved.

Optimal solution: $(x_1, x_2) = (1, 4) \qquad Value = 9.$