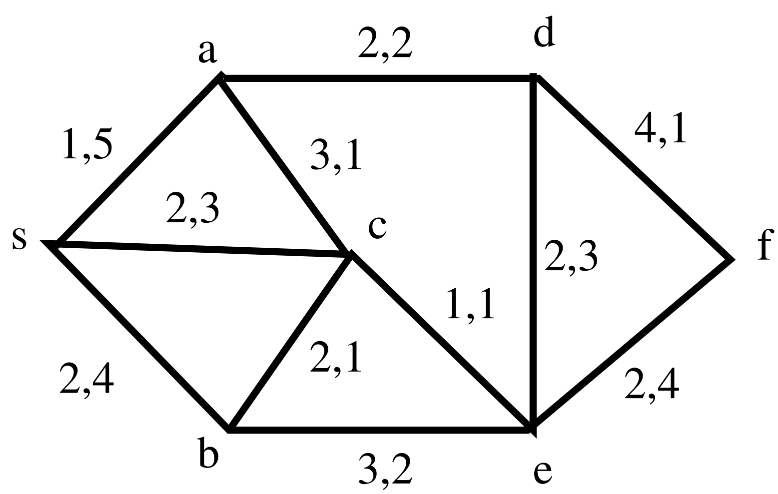
Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

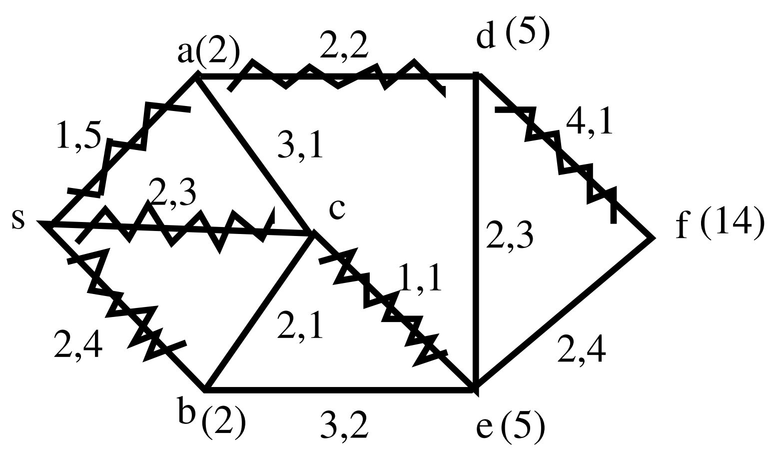
OPERATIONS RESEARCH II 21-393

Homework 2: Due Friday September 19.

1. Find a shortest path from s to all other nodes in the digraph below. Each edge (x, y) is labelled by a pair (a, b) and the length of the corresponding arc is $a + bt$ where t is the time the path reaches x. All arcs are directed lexicograhically e.g. (c, e) is directed from c to e.



Solution



2. Let $\mathcal{W}$ denote the set of walks in a directed graph D. If W_1 is a walk from a to b and W_2 is a walk from b to c then W_1+W_2 is the walk from a to c obtained by following W_1 and then W_2.

   Let $\ell : \mathcal{W} \to \mathbb{R}$ be a real valued function defined on $\mathcal{W}$. Suppose that it has the following properties:

   (a) $\ell(C) \geq 0$ for any closed walk C. (A walk is closed if it begins and ends at the same vertex).

   (b) If W_1, W_1' are walks from a to b and W_2, W_2' are walks from b to c and $\ell(W_i') \geq \ell(W_i)$ for $i = 1, 2$ then $\ell(W_1' + W_2') \geq \ell(W_1 + W_2)$.

   Consider the following algorithm: n is the number of vertices in D.
   Initialise $W_{i,j} = (i, j)$ and $D_{i,j} = \ell(W_{i,j})$ for $i, j = 1, 2, \ldots, n$.

   For $k = 1$ to n **Do**
       For $i = 1$ to n **Do**
           For $j = 1$ to n **Do**
           $D_{i,j} \leftarrow \min\{D_{i,j}, \ell(W_{i,k} + W_{k,j})\}$
           If $D_{i,j} = \ell(W_{i,k} + W_{k,j})$ **then** $W_{i,j} \leftarrow W_{i,k} + W_{k,j}$
           oD
       oD
   oD

   Prove that when the algorithm finishes,

   $$D_{i,j} = \min\{\ell(P) : P \text{ is a path from } i \text{ to } j\}.$$

   Solution: We argue by induction on k that at the end of k executions of the outermost loop, for all i, j, $D_{i,j}$ minimises $\ell(P)$ over all walks from i to j whose interior vertices are in $\{1, 2, \ldots, k\}$.

   This is trivially true for $k = 0$, since the claim is about the "lengths" of the edges (i, j).

   Assume that the claim is true for $k \geq 0$. A walk from i to j either uses vertex $k+1$ in its interior or it doesn't. By induction, the shortest walk from i to j that doesn't use $k+1$ is $D_{i,j}$. So, we only have to argue that $\ell(W_{i,k+1} + W_{k+1,j})$ is the length of a shortest walk from i to k that uses $k+1$.

Let $W = W_1 + W_2 + W_3$ be a walk from i to j where
W_1 goes from i to $k+1$ and only uses $\{1, 2, \ldots, k\}$ in its interior;
W_2 goes from $k+1$ to $k+1$;
W_3 goes from $k+1$ to j and only uses $\{1, 2, \ldots, k\}$ in its interior.

It follows from Property (b) that

$$\ell(W_1 + W_2) \geq \ell(W_1 + \Lambda) = \ell(W_1)$$

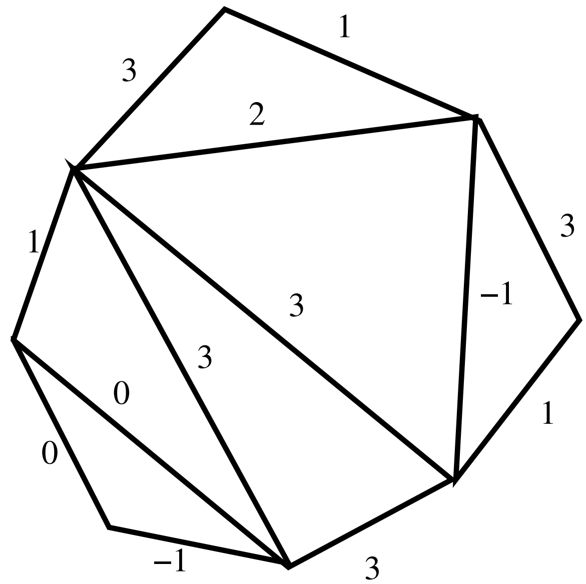
where Λ is the path from $k+1$ to $k+1$ that consists of the single vertex $k+1$.

Applying (b) again, we see that

$$\ell(W) \geq \ell(W_1 + W_3) \geq \ell(W_{i,k+1} + W_{k+1,j}).$$

This completes the induction. So, for each i, j, $W_{i,j}$ minimises $\ell(W)$ over all walks from i to j. Now if $W_{i,j}$ is not a path, then we can write it as $W_1 + W_2 + W_3$ as above and show that there is a walk W' from i to j with $\ell(W') \leq \ell(W_{i,j})$ and with fewer edges. Clearly $\ell(W') = \ell(W_{i,j})$ here, but then we have that the walk from i to j that minimises ℓ and has fewest edges among walks that minimise ℓ, must be a path.

3. Find a minimum weight spanning tree in the weighted graph below:



Solution