10/22/14

Two Person Zero Sum Games

$$\begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix}$$

2 players A, B

$m \times n$ matrix $A = [a_{ij}]$

simultaneous $\left\{ \begin{array}{l} A \text{ chooses } i \\ B \text{ chooses } j \end{array} \right.$ and then B pays A a_{ij}

$$\begin{bmatrix} 4 & -3 & 2 \\ 3 & -1 & 4 \\ 2 & 5 & -3 \end{bmatrix}$$

A plays $\underline{1}$
B plays 2 $\rightarrow$ B pays A -3 $\equiv$ A pays B 3

Ex: Rock-Paper-Scissors

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

Tennis

Receiver   guess

Server

$$\begin{array}{cc} S\backslash R & \begin{array}{ccc} M & F & S \end{array} \\ \begin{array}{c} M \\ F \\ S \end{array} & \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \end{array}$$

$S_A = \{\text{strategies for } A\} \leftarrow \text{currently} = \{1, 2, \cdots, m\}$

$S_B = \{\text{strategies for } B\} \leftarrow \quad \text{"} \quad = \{1, 2, \cdots, n\}$

$(u, v) \in S_A \times S_B$ then payoff $= PAY(u, v)$.

$(u_0, v_0) \in S_A \times S_B$ is a stable solution

(Nash Equilibrium)

$$PAY(u, v_0) \leq PAY(u_0, v_0) \leq PAY(u_0, v)$$

$\forall u \in S_A$ $\qquad$ $\forall v \in S_B$

Neither player has an incentive to change

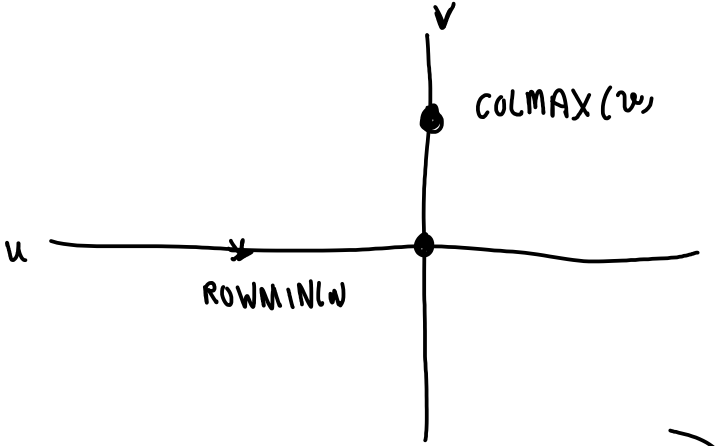
When is there a stable solution?

$$\mathrm{ROWMIN}(u) = \min_{v \in S_B} \mathrm{PAY}(u,v) \qquad P_A = \max_u \mathrm{ROWMIN}(u)$$

$$\mathrm{COLMAX}(v) = \max_{u \in S_A} \mathrm{PAY}(u,v) \qquad P_B = \min_v \mathrm{COLMAX}(v)$$

(i) $P_A \leq P_B$

(ii) $S_A \times S_B$ contains a stable solution iff $P_A = P_B$



$\forall\, u, v$

$$\mathrm{ROWMIN}(u) \leq \mathrm{PAY}(u, v) \leq \mathrm{COLMAX}(v)$$

$$\Rightarrow P_A \leq P_B$$

Stability

① Suppose (u_0, v_0) is stable

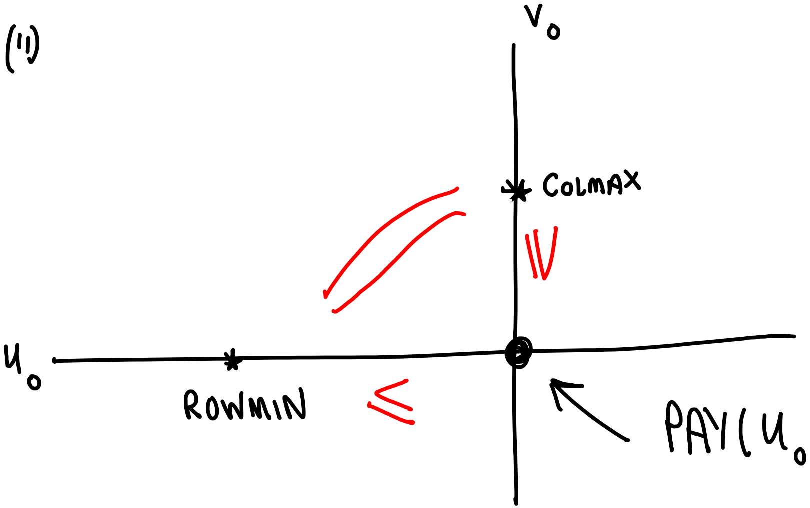
$$PAY(u, v_0) \le PAY(u_0, v_0) \le PAY(u_0, v)$$

$PAY(u_0, v_0) = COLMAX(v_0)$

$PAY(u_0, v_0) = ROWMIN(u_0)$

$$\forall u \quad ROWMIN(u) \le \overset{= P_A}{ROWMIN(u_0)} = PAY(u_0, v_0) = \overset{P_B}{COLMAX(v_0)} \le COLMAX(v), \ \forall v$$

(11)



$P_A = \mathrm{ROWMIN}(u_0)$

$= P_B$

$= \mathrm{COLMAX}(v_0)$

$\mathrm{PAY}(u_0, v_0)$ is stable

$$\begin{array}{c|ccc} & R & P & S \\ \hline R & & & \\ P & & \left[\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{matrix}\right] & \\ S & & & \end{array}$$

$$\begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{matrix} R & P & S \end{matrix} \\ \left[\begin{matrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{matrix}\right] \end{array}$$

$P_A = -1$

$P_B = 1$

NO STABLE SOLUTION IF WE USE "PURE" STRATEGIES