10/1/14

Set Covering

Given $S_1, S_2, \dots, S_n \subseteq \{1, 2, \dots, m\} = [m]$

Each set S_j has a cost c_j

A cover $I \subseteq \{1, 2, \dots, n\}$ satisfies $\bigcup_{i \in I} S_i = [m]$

Cost $c(I) = \sum_{i \in I} c_i$.

Problem: find minimum cost cover.

$$x_j = \begin{cases} 0 & \text{do not included } S_j \\ 1 & \text{include } S_j \end{cases}$$

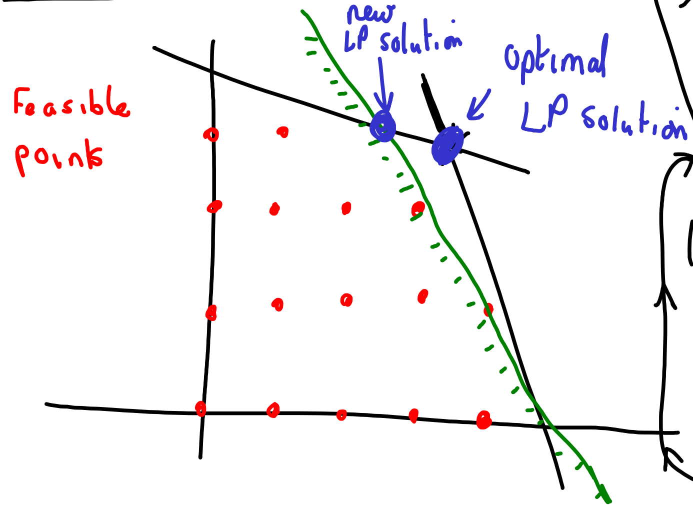
$$\text{Minimise cost} = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

$$a_{ij} = \begin{cases} 0 & i \notin S_j \\ 1 & i \in S_j \end{cases}$$

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n \geq 1 \quad , \quad i = 1, 2, \dots, m$$

$$x_j = 0 \text{ or } 1$$

Cutting Plane Algorithm



Idea: (i) Solve the LP relaxation ie ignore integrality

If solution is integer feasible – done. IP is solved

(ii) If not, add a cut – constraint that cuts off LP solution but no integer solution.

Gomory Cuts: pure integer programs.

Suppose that we know

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

$x_1, x_2, \ldots x_n$ are non-negative integers

this will be an equation from the optimal LP tableau if we do not have an integer solution

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Write $a_j = \lfloor a_j \rfloor + f_j$ $\qquad$ $b = \lfloor b \rfloor + f$

integer part $\qquad$ fractional part

$$5\tfrac{1}{3} = 5 + \tfrac{1}{3}$$

$$-5\tfrac{1}{3} = -6 + \tfrac{2}{3}$$

$$\sum_{j=1}^{n} (\lfloor a_j \rfloor + f_j) x_j = \lfloor b \rfloor + f$$

$$\Downarrow$$

$$\sum_{j=1}^{n} f_j x_j - f = \overbrace{\lfloor b \rfloor - \sum_{j=1}^{n} \lfloor a_j \rfloor x_j}^{\text{Integer}}$$

Now remember that $x_1 x_2, \ldots x_n \geq 0$ & integer.

LHS is an integer & $\geq -f > -1$ so LHS ≥ 0.

Suppose that

$$\underbrace{x_i}_{\text{basic variable}} + \sum_{j \in J \leftarrow \text{non-basics}} b_{ij} x_j = b_{i0} \leftarrow \text{not integer}$$

is the i th row of simplex tableau

Every feasible solution satisfies

$$\sum_{j \in J} f_j x_j \geq f > 0$$

Maximise $\quad 3x_1 + 4x_2$

s.t.

$$x_1 + 2x_2 \leq 4$$

$$2x_1 + x_2 \leq 6$$

$$x_1, x_2 \geq 0 \text{ \& integer}$$

bv	x_1	x_2	x_3	x_4	RHS
x_0	-3	-4			0
x_3	1	2	1		4
x_4	2	1		1	6