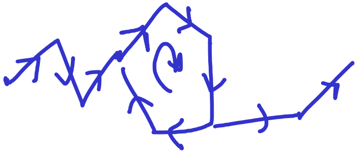
9/15/14

Now we allow arc lengths to be negative.

Suppose however $\exists$ a cycle C such $\ell(C) < 0$.

This means that: there is no lower bound to the length of walks.



go round cycle k times $\ell(W) = \alpha - k\,\ell(C)$

$\to -\infty$

We study shortest paths when we allow $\ell(e) < 0$ but we do not allow $\ell(C) < 0$ for a directed cycle C.

Claim: length of shortest walk from a to b = length of shortest path from a to b

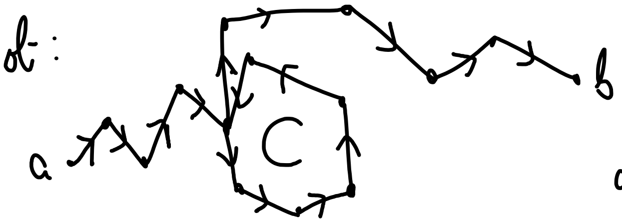
Proof of claim

Let W have fewest edges among all shortest walks from a to b.

Then W is a path.

Suppose not:

W:



$W' = W - C$ is no longer and has fewer edges

Optimality criterion:

Suppose we have a set of walks $\{W_v\}$ from s to v for all vertices v.

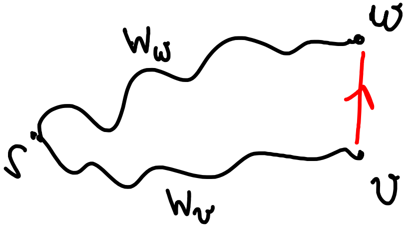
$d(v) = \ell(W_v)$.

These walks are all shortest walks iff

(*) $\quad d(w) \leq d(v) + \ell(v,w)$ for all v, w.

Proof

(*) does not hold and $d(w) > d(v) + \ell(v,w)$



$\Rightarrow$ $W_v + (v,w)$ is shorter than W_w

Suppose (*) does hold.

Fix s and v and let $W=(s=v_0, v_1, \dots, v_k=v)$ be a walk from s to v.

add up inequalities

$$\left\{\begin{aligned} d(v_1) - d(\overset{0}{v_0}) &\leq \ell(v_0, v_1) && * \\ d(v_2) - d(v_1) &\leq \ell(v_1, v_2) && * \\ &\vdots \\ d(v_k) - d(v_{k-1}) &\leq \ell(v_{k-1}, v_k) && * \end{aligned}\right.$$

$\Rightarrow d(v_k) \leq \ell(W)$

Algorithm $D = (V, E = \{e_1, e_2, \dots, e_m\})$

Initialise : $d(s) = 0$ & $d(v) = l(s,v)$ length of edge (s,v)

All walks W_v are single edge (s,v) initially

repeat

for $i = 1$ to m do

$e_i = (x_i, y_i)$; if $d(y_i) > d(x_i) + l(x_i, y_i)$

then put $d(y_i) = d(x_i) + l(x_i, y_i)$

Replace W_{y_i} by $W_{x_i} + (x_i, y_i)$

od

until $(*)$ holds

<u>Claim:</u> Algorithm finishes in at most $n-1$ iterations $[n = |V|]$

<u>Proof</u>

Let $V_k = \{ v : \exists$ a shortest path from s to v using $\leq k$ edges $\}$

$\boxed{V = V_{n-1}}$

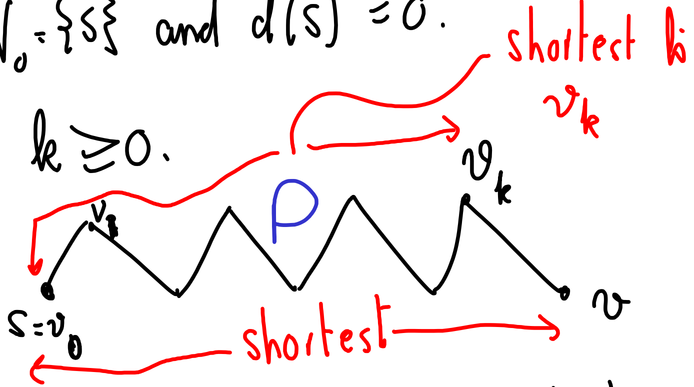
Claim: after k iterations $d(v)$ is correct for $v \in V_k$.

Proof is by induction on k.

Base Case: $k=0$; $V_0=\{s\}$ and $d(s)=0$.

Assume true for some $k \geq 0$.

$v \in V_{k+1} / V_k$



After $k+1$ iterations $d(v) \leq d(v_k) + \ell(v_k, v) = \ell(P)$