9\9\13

Infinite time horizon

Production problem going on indefinitely.

Consider some cost sequences

A	5	5	5	5	5	. . . .
B	4	4	4	4	4	. . .
C	3	5	3	5	3	. . .

B is better than A

C is better than B with inflation

Discounted Cash Flow or Net Present Value.

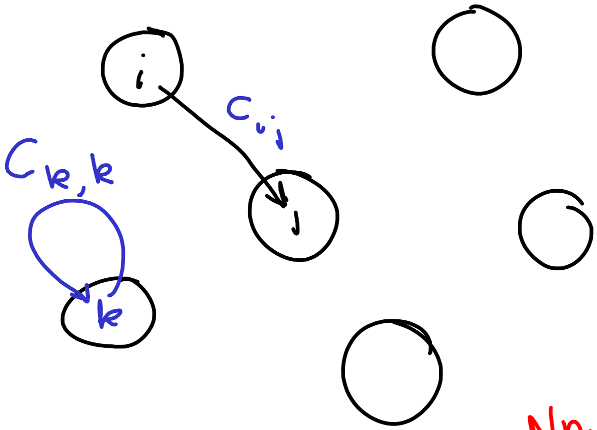
We assume a discount rate of α $\alpha < 1$

$C_1, C_2, C_3, \quad .$

gives

NPV of $C_1 + \alpha C_2 + \alpha^2 C_3 + \ldots$

System of n states:



c_{ij} = cost of going from state i to state j

If you through states $i_1, i_2, i_3, i_4, \ldots$

$$NPV = c(i_1,i_2) + \alpha c(i_2,i_3) + \alpha^2 c(i_3,i_4)$$

Find a policy that minimises NPV.

Suppose optimum sequence is

$$i_1^*,\ i_2^*,\ i_3^*,\ \ldots\ldots,\ i_k^*,\ i_{k+1}^*\ \ldots\ldots\ i_l^*\quad i_{l+1}^*$$

$$i_k^* = i,\quad i_{k+1}^* = j,\quad i_l^* = i,\quad i_{l+1}^* = j$$

Proceeding optimally from $i \to j$ next

So, $\exists\ \pi : [n] \longrightarrow [n]$

so that when at i, go next to $\pi(i)$

There are n^n choices for π.

Evaluating π.

y_i = NPV of following π indefinitely

$$= c_{i,\pi(i)} + \alpha\, y_{\pi(i)}$$

$$c_{i,\pi(i)} + \alpha\, c_{\pi(i),\pi^2(i)} + \alpha^2 c_{\pi^2(i),\pi^3(i)} + \cdots$$

$$C = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 3 & 4 & 1 \\ 2 & 4 & 1 & 3 \\ 5 & 4 & 3 & 4 \end{bmatrix} \qquad \alpha = 1/2$$

Start with

i	1	2	3	4
$\pi(i)$	4	3	2	1

Algorithm

Current π

Evaluate

Check for optimal

NO: CHANGE π

yes

DONE

$$C = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 3 & 4 & 1 \\ 2 & 4 & 1 & 3 \\ 5 & 4 & 3 & 4 \end{bmatrix}$$

$$y_1 = 5 + \frac{1}{2} y_4 = 10$$

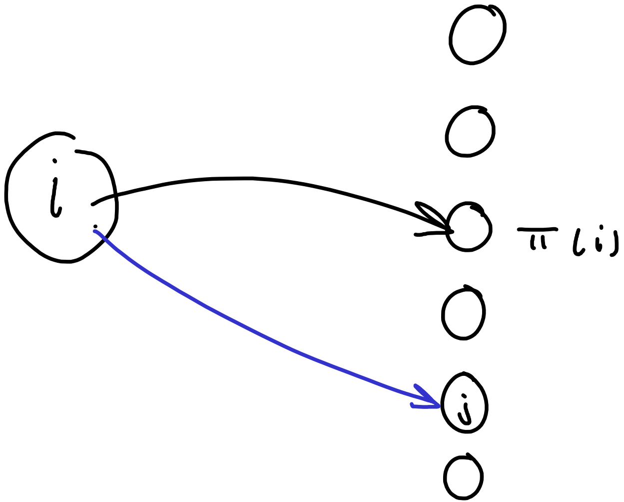
$$y_2 = 4 + \frac{1}{2} y_3 = 8$$

$$y_3 = 4 + \frac{1}{2} y_2 = 8$$

$$y_4 = 5 + \frac{1}{2} y_1 = 10$$

We can find a π that simultaneously minimises $y_1, y_2, \ldots$

Optimality condition



$c_{i,\pi(i)} + \alpha y_{\pi(i)}$

$c_{ij} + \alpha y_j$

$i \to j$ then follow π.

Check

$$y_i = \min_j \left[c_{i,j} + \alpha y_j \right]$$

Claim: true iff π simultaneously minimizes $y_1, y_2, \ldots, y_n$.

Check:

$\underline{i = 1}$

$$\begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \\ 10 & 8 & 8 & 10 \end{array}$$

$$C = \begin{bmatrix} 3 & 2 & 1 & 5 \\ 4 & 3 & 4 & 1 \\ 2 & 4 & 1 & 3 \\ 5 & 4 & 3 & 4 \end{bmatrix}$$

$$3 + \frac{1}{2} y_1 = 8$$

$$2 + \frac{1}{2} y_2 = 6$$

$$1 + \frac{1}{2} y_3 = 5^*$$

$$5 + \frac{1}{2} y_4 = 10$$