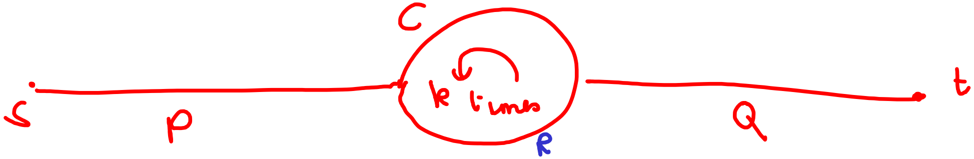
9/25/13

Shortest paths with some negative arc lengths.

We replace problem by that of finding a shortest walk from s to all other vertices.

Problem: Suppose $\exists$ cycle C with $\ell(C) = \lambda < 0$

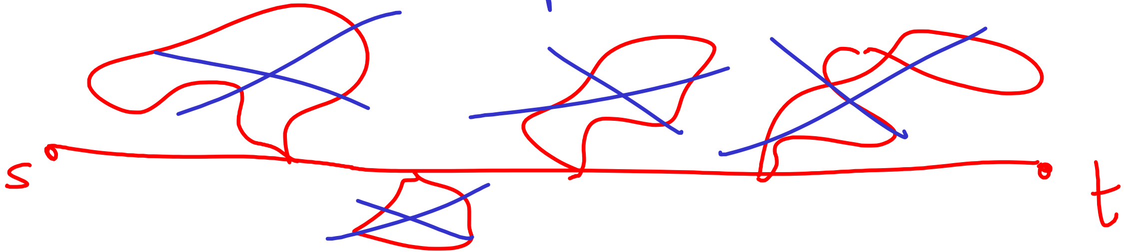


length $= \ell(P) + k\lambda + \ell(Q) + \ell(R) \to -\infty$ as $k \to \infty$.

We assume that there are no negative cycles

i.e $\ell(C) \geq 0$ for all directed cycles.

Now a shortest path is also a shortest walk



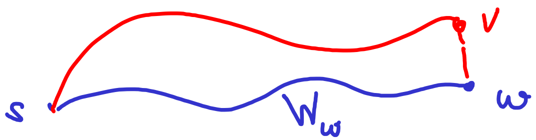
Optimality condition:

Suppose we have walks W_v, $v \in V$ of length $d(v)$.

These are all shortest walks iff

$$(*) \qquad d(w) \le d(v) + \ell(v,w) \quad \forall (v,w) \in \text{Edges}$$

Suppose $d(w) > d(v) + \ell(v,w)$.



Red < Blue

Suppose $\circledast$ holds

Fix v and take a walk W from s to

$$W = (s = v_0, v_1, v_2, \dots, v_k = v)$$

Need to show that $\ell(W) \geq d(v)$

$$\overset{d(v)}{d(v_k)} \overset{-}{\leq} d(v_{k-1}) \overset{\leq}{+} \ell(v_{k-1}, v_k)$$

$$d(v_{k-1}) \overset{-}{\leq} d(v_{k-2}) \overset{\leq}{+} \ell(v_{k-2}, v_{k-1})$$

$$\vdots$$

$$d(v_1) \overset{-}{\leq} \overset{=0}{d(s)} \overset{\leq}{+} \ell(v_0, v_1)$$

Add up inequalities : $d(v) \leq \ell(W)$

Algorithm

Start with some walks. $W_v, v \in V$

$$d(v) = \ell(W_v), \ \forall v$$

repeat

if $\exists v, w$ s.t $d(w) > d(v) + \ell(v,w)$ then

replace W_w by $W_v + (v,w)$:

until optimality condition.

Finite Termination

$\sum_v d(v)$ strictly decrease at each step.

Can never repeat the same set of walks.

We can always remove cycles in walks so that we always have a set of paths.

# of sets of paths is finite.

Here is an $O(mn)$ time version. m = # edges

n = # vertices

Simple Algorithm

$E = \{e_1, e_2, \ldots, e_m\}$ $\quad e_i = (x_i, y_i)$

ROUND

repeat

for $i = 1$ to m do

if $d(y_i) > d(x_i) + l(x_i, y_i)$

$d(y_i) \leftarrow d(x_i) + l(x_i, y_i)$

$W_{y_i} \leftarrow W_{x_i} + (x_i, y_i)$

until optimal

* after checking the edges $d(w)$ is correct

Terminates in $\leq n$ rounds

After k round, $d(v)$ is correct for every v for which there is a shortest walk (path) using $\leq k$ edges.