9\23\13

Shortest Paths

Directed graph $D = (V, A)$ and every arc a has a length $l(a)$.

Length of path $P = (x_0, x_1, \dots, x_k)$

to $l(x_0, x_1) + l(x_1, x_2) + \dots + l(x_{k-1}, x_k)$

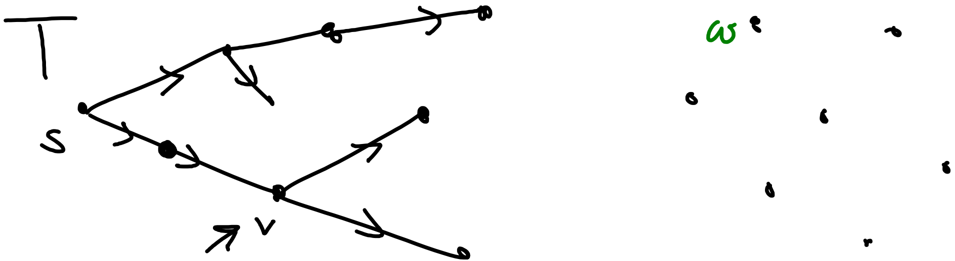
Problem: given vertices x, y find shortest path from x to y.

Case 1: $\ell(a) \geq 0$.

Dijkstra's algorithm.

Fixed vertex s - find shortest path s to all other vertices.

At some stage we have a tree T

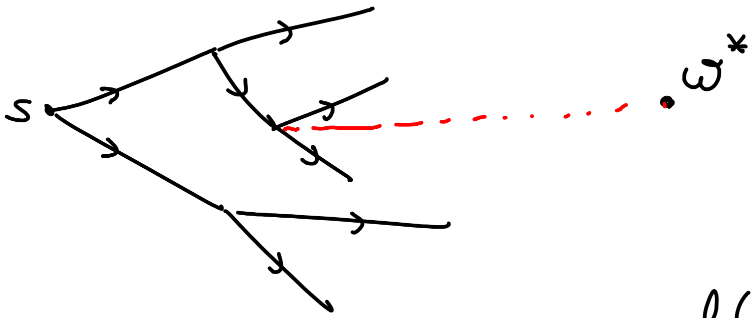


$d(w)$ = min. length path $s \to w$ that goes $\underbrace{s \to v}_{\subseteq T} \to w$

$d(w^*) = \min_{w \notin T} d(w)$

$d(v)$ = length of the T-path $s \to v$. Claim: $d(v)$ = length of shortest

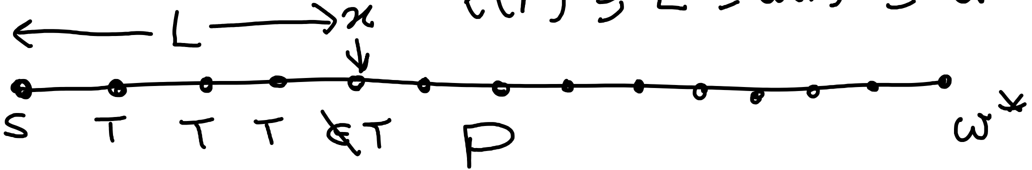
Claim: $d(w^*)$ = length of shortest path to w^*



Dijkstra works for any definition of length for which $\ell(P) \geq \ell(Q)$ where

only place where non-negativity is used.

$$\ell(P) \geq L \geq d(x) \geq d(w^*)$$



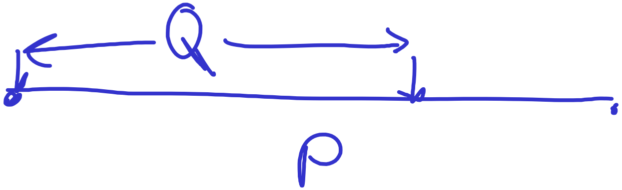
Time dependent arc length.

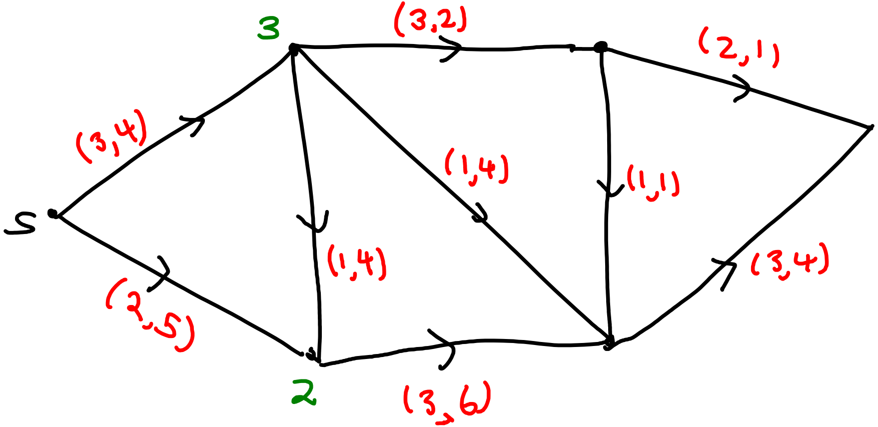
Suppose length of $\underset{(x,y)}{\overset{\shortparallel}{a}}$ depends on time of arrival at x.

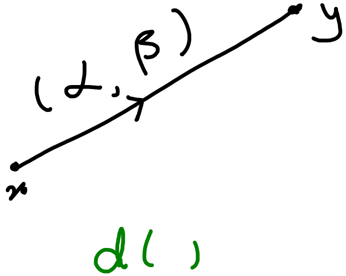
For example: assume $\ell(a,t) = \alpha_a + \beta_a t$

where $\alpha_a, \beta_a \geq 0$.

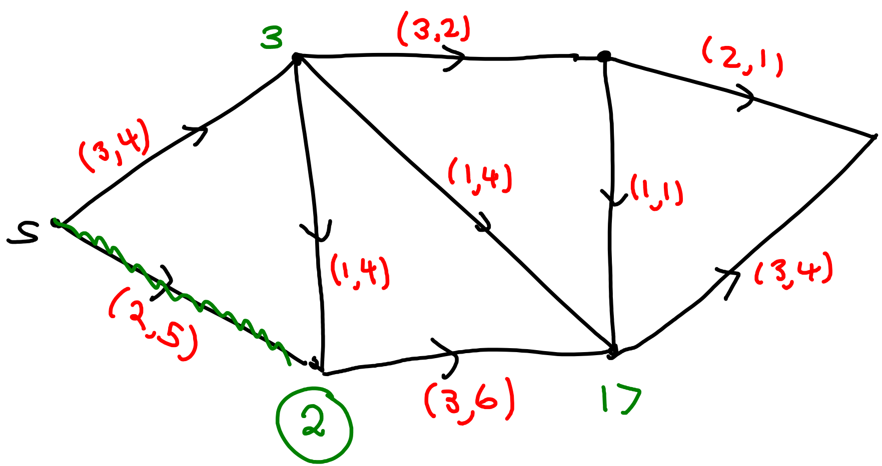
So $\ell(P) \geq \ell(Q)$



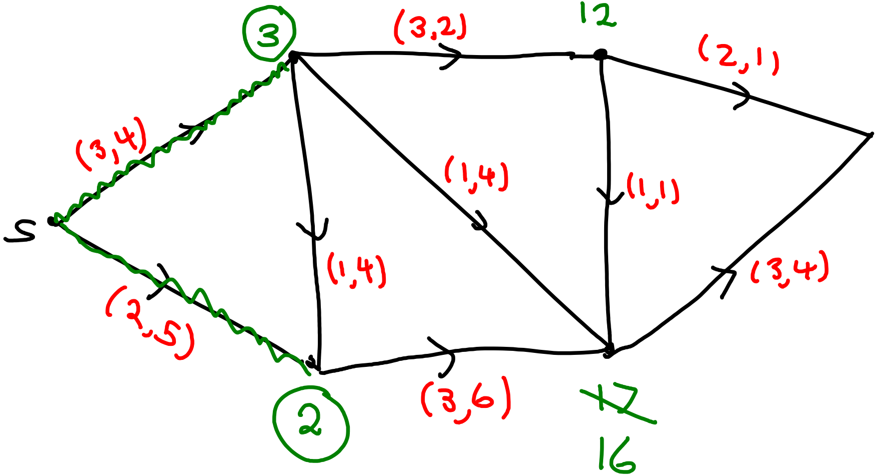




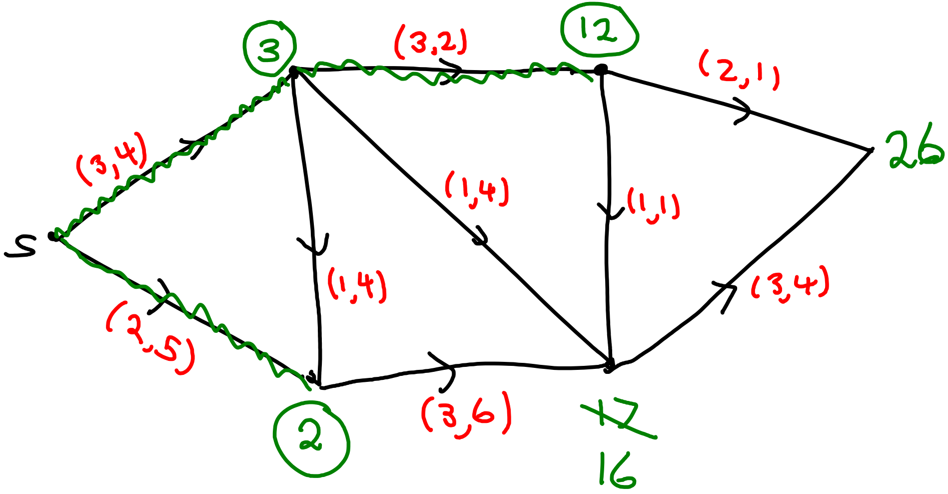
Start at $t=0$.



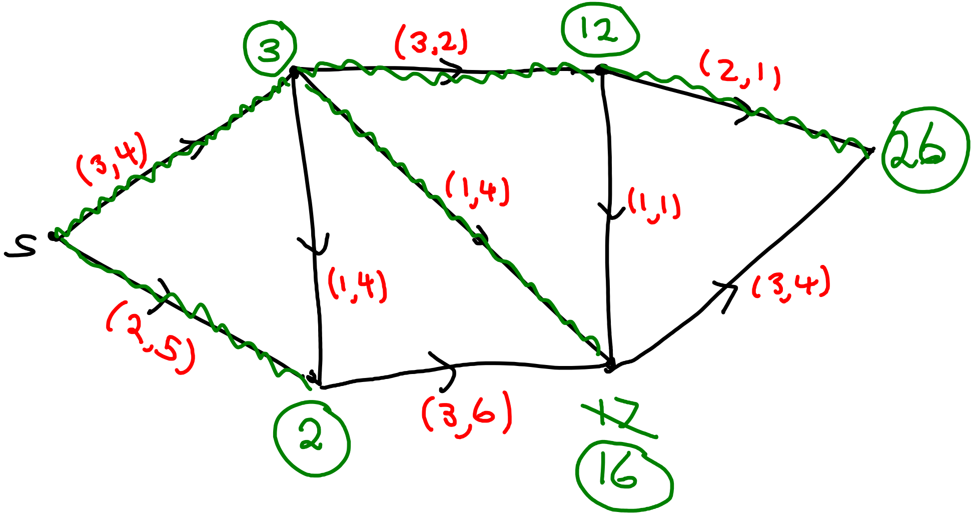
s
3
2
17
(3,4)
(2,5)
(1,4)
(3,2)
(1,4)
(3,6)
(1,1)
(2,1)
(3,4)



s
3
2
12
~~17~~
16
(3,4)
(2,5)
(1,4)
(3,2)
(1,4)
(3,6)
(2,1)
(1,1)
(3,4)



s
3
12
2
26
~~17~~
16
(3,4)
(2,5)
(3,2)
(2,1)
(1,4)
(1,4)
(1,1)
(3,6)
(3,4)



s
3
12
26
2
~~17~~
16
(3,4)
(2,5)
(1,4)
(3,2)
(1,4)
(3,6)
(2,1)
(1,1)
(3,4)

Now suppose that some arc lengths are negative.

Finding a shortest path from x to y, when there are negative length arcs is NP hard. — it is as hard as TSP, Integer Programming. . . .

Hamilton Path Problem: Given x, y; does there exist a path from x to y that goes through every vertex.
[Only visit a vertex once] NP-HARD

Suppose one can solve shortest path x to y when there are negative arc lengths in time $T(n)$: n = # of vertices

P = program