10\23\13

$$\begin{array}{cc} A\backslash B & v \\ u & \begin{bmatrix} & & \\ & \mathrm{PAY}(u,v) & \\ & & \end{bmatrix} \end{array}$$

Expected payoff

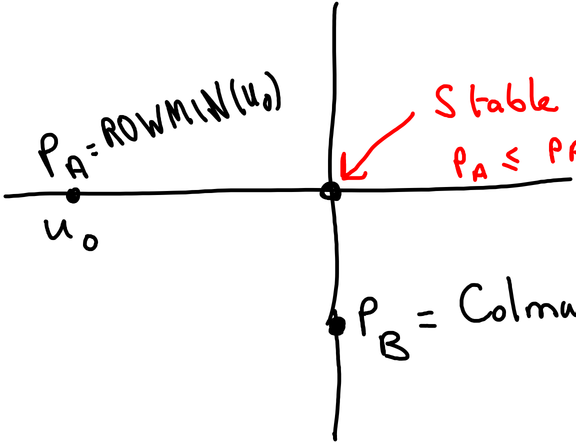
$$p_A = \max_u \ \mathrm{ROWMIN}(u)$$

$$p_B = \min_v \ \mathrm{COLMAX}(v)$$

Claim $p_A \le p_B$ &

$p_A = p_B$ iff $\exists$ stable solution

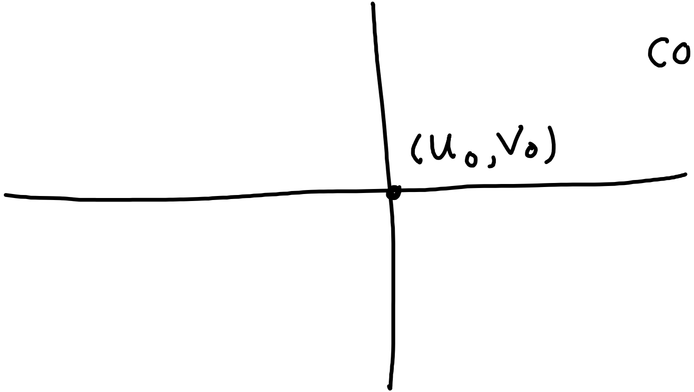
Suppose $P_A = P_B$



Stable solution

$P_A \leq PAY(u_0, v_0) \leq P_B \Rightarrow PAY(u_0, v_0) = P_A = P_B$

Suppose that (u_0, v_0) is stable



STABLE $\Longrightarrow$

$$\begin{array}{ccc} COLMAX(v_0) = & PAY(u_0, v_0) & = ROWMIN(u_0) \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\geq$} \\ p_B & \geq & p_A \end{array}$$

Not all matrices have stable solutions

$$\begin{array}{c|ccc} & R & P & S \\ \hline R & 0 & -1 & +1 \\ P & +1 & 0 & -1 \\ S & -1 & +1 & 0 \end{array}$$

$P_A = -1$

$P_B = +1$

<u>Pure</u> strategy (i) – play i repeatedly

No <u>pure</u> stable solution.

Need <u>mixed</u> strategies

A mixed strategy for A is a probability vector $(p_1, p_2, \dots, p_m)$ where $p_i \geq 0$, $i = 1, 2, \dots, m$

and $p_1 + p_2 + \dots + p_m = 1$

A plays ~~row~~ i with probability p_i

A mixed strategy for B is $q_1, q_2, \dots, q_n \geq 0$

where $q_1 + \dots + q_n = 1$.

$$\begin{array}{cc} A \backslash B & \underline{q} \\ \underline{p} & \left[\; PAY(\underline{p}, \underline{q}) \; \right] \end{array}$$

$$PAY(\underline{p}, \underline{q}) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \, p_i \, q_j$$

$$P_A = \max_{\underline{p}} \min_{\underline{q}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

$$= \max_{\underline{p}} \left(\min_{\underline{q}} \sum_{j=1}^{n} c_j(p)\, q_j \right)$$

$$c_j(p) = \sum_{i=1}^{m} a_{ij} p_i$$

$$= \max_{\underline{p}} \min_{j} \{ c_1(p), c_2(p), \ldots, c_n(p) \}$$

$$\max_{\underline{p}} \min_{j} \{ c_1(p), c_2(p), \dots, c_n(p) \}$$

$$= \text{maximum } \xi$$

$$\xi \leq c_1(p) = \sum_{i=1}^{m} a_{i1} p_i$$

$$\xi \leq c_2(p) = \sum_{i=1}^{m} a_{i2} p_i$$

$$\vdots$$

$$p_1 + p_2 + \cdots + p_m = 1, \quad p_i \geq 0 \quad i = 1, 2, \dots, m.$$

$$P_A = \max \xi$$

$$j = 1, \dots, m: \quad \xi \leq \sum_{i=1}^{m} a_{ij} p_i$$

$$\sum_{i=1}^{m} p_i = 1$$

$$i = 1, 2, \dots, m \quad p_i \geq 0$$

$$P_B = \min \eta$$

$$\eta \geq \sum_{j=1}^{n} a_{ij} q_j \quad i = 1, 2, \dots, m$$

$$\sum_{j=1}^{n} q_j = 1$$

$$q_j \geq 0, \quad j = 1, 2, \dots, n$$

The two linear programs are dual to each other

Both LP's are feasible.