Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 2: Due Monday, September 24.

Q1 Consider the simple production problem of the notes with costs $c(x)$, demands d_t, $t = 1, 2, \ldots, n$ and maximum inventory H and a penalty π per period for each order still unfilled. If you decide that you want to make x, then you will actually make $x-1, x$ or $x+1$ with probability 1/3 each. (Making -1 should be taken to mean making 0).
Formulate a Dynamic Programming recurrence to solve the problem of minimising expected total cost.
Solution: Let $f_r(i)$ be the minimum expected cost of fulfilling demand for periods $t, t+1, \ldots, n$. Then we have

$$f_t(i) =$$
$$\min_x \left\{ \frac{1}{3} \sum_{\theta=0,\pm1} (c((x+\theta)^+) + \pi(d_t - (i+x+\theta))^+ + f_{t+1}((i+x+\theta-d_t)^+)) \right\}$$

where $f_t(i) = f_t(H)$ if $i > H$.
Q2 A system can be in 3 states 1,2,3 and the cost of moving from state i to state j in one period is $c(i,j)$, where the $c(i,j)$ are given in the matrix below. The one period discount factor α is 1/2.
The aim is to find a policy which simultaneously minimises the discounted cost of operating from any starting state. Start with the policy

$$\pi(1) = 1, \pi(2) = 3, \pi(3) = 2.$$

Find an optimal policy.
The matrix of costs is

$$\begin{bmatrix} 10 & 2 & 3 \\ 5 & 2 & 8 \\ 1 & 8 & 2 \end{bmatrix}$$

Solution: Start with an initial solution of $\pi(1) = 1, \pi(2) = 1, \pi(3) = 2$.

Then we solve

$$\begin{aligned} y_1 &= 10 + \frac{y_1}{2} \\ y_2 &= 8 + \frac{y_1}{2} \\ y_3 &= 8 + \frac{y_2}{2} \end{aligned}$$

Solving these equations gives $y_1 = 20, y_2 = 18, y_3 = 17$.
We now check for optimality:
Check y_1:

$$\begin{aligned} 10 &+ \frac{y_1}{2} = 20 \\ 2 &+ \frac{y_2}{2} = 11 \quad \star \\ 3 &+ \frac{y_3}{2} = \frac{23}{2} \end{aligned}$$

Check y_2:

$$\begin{aligned} 5 &+ \frac{y_1}{2} = 15 \\ 2 &+ \frac{y_2}{2} = 11 \quad \star \\ 8 &+ \frac{y_3}{2} = \frac{33}{2} \end{aligned}$$

Check y_3:

$$\begin{aligned} 1 &+ \frac{y_1}{2} = 11 \\ 8 &+ \frac{y_2}{2} = 17 \\ 2 &+ \frac{y_3}{2} = \frac{21}{2} \quad \star \end{aligned}$$

Our new policy is $\pi(1) = 2, \pi(2) = 2, \pi(3) = 3$.

Then we solve

$$\begin{aligned} y_1 &= 2 + \frac{y_2}{2} \\ y_2 &= 2 + \frac{y_2}{2} \\ y_3 &= 2 + \frac{y_3}{2} \end{aligned}$$

Solving these equations gives $y_1 = 4, y_2 = 4, y_3 = 4$.
We now check for optimality:
Check y_1:

$$\begin{aligned} 10 + \frac{y_1}{2} &= 12 \\ 2 + \frac{y_2}{2} &= 4 \quad \star \\ 3 + \frac{y_3}{2} &= 5 \end{aligned}$$

Check y_2:

$$\begin{aligned} 5 + \frac{y_1}{2} &= 7 \\ 2 + \frac{y_2}{2} &= 4 \quad \star \\ 8 + \frac{y_3}{2} &= 10 \end{aligned}$$

Check y_3:

$$\begin{aligned} 1 + \frac{y_1}{2} &= 3 \quad \star \\ 8 + \frac{y_2}{2} &= 10 \\ 2 + \frac{y_3}{2} &= 4 \end{aligned}$$

Our new policy is $\pi(1) = 2, \pi(2) = 2, \pi(3) = 1$.

Then we solve

$$\begin{aligned}
y_1 &= 2 + \frac{y_2}{2} \\
y_2 &= 2 + \frac{y_2}{2} \\
y_3 &= 1 + \frac{y_1}{2}
\end{aligned}$$

Solving these equations gives $y_1 = 4, y_2 = 4, y_3 = 3$.
We now check for optimality:
Check y_1:

$$\begin{aligned}
10 &+ \frac{y_1}{2} = 9 \\
2 &+ \frac{y_2}{2} = 4 \quad \star \\
3 &+ \frac{y_3}{2} = \tfrac{13}{2}
\end{aligned}$$

Check y_2:

$$\begin{aligned}
5 &+ \frac{y_1}{2} = 7 \\
2 &+ \frac{y_2}{2} = 4 \quad \star \\
7 &+ \frac{y_3}{2} = \tfrac{17}{2}
\end{aligned}$$

Check y_3:

$$\begin{aligned}
1 &+ \frac{y_1}{2} = 3 \quad \star \\
8 &+ \frac{y_2}{2} = 10 \\
2 &+ \frac{y_3}{2} = \tfrac{7}{2}
\end{aligned}$$

So the current policy is optimal.

Q3 You are an oil trader and the price of oil fluctuates in the following way:

If the price per barrel in period i is p_i dollars then

$$p_{i+1} = \begin{cases} (1+\varepsilon)p_i & \textit{Probability } p \\ (1-\varepsilon)p_i & \textit{Probability } 1-p \end{cases}$$

The current price is one hundred dollars per barrel. Give a dynamic programming formulation to compute the maximum price per barrel one should pay now for the following option: Purchase oil at price c dollars per barrel, at any time $t = 1, 2, \ldots, n$.
Explanation: Here one is paying for the (hoped for) opportunity to purchase the oil cheaply in the future.
Hint: Let $f_t(a)$ be the maximum expected value of the option at time t if the current price is a.
Solution:

$$f_t(a) = \max\{a - c,\, pf_{t+1}((1+\varepsilon)a) + (1-p)f_{t+1}((1-\varepsilon)a)\}$$

and $f_{n+1}(a) = 0$ for all a.
You should pay no more than $f_0(100)$.