Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 1: Due Monday September 10.

Q1 Solve the following knapsack problem:

$$\begin{array}{llllllll} \text{maximise} & 4x_1 & + & 8x_2 & + & 15x_3 & & \\ \text{subject to} & & & & & & & \\ & 3x_1 & + & 4x_2 & + & 5x_3 & \leq & 19 \end{array}$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution

w	f_1	δ_1	f_2	δ_2	f_3	δ_3
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	4	1	4	0	5	0
4	4	1	8	1	8	0
5	4	1	8	1	15	1
6	8	1	10	0	15	1
7	8	1	13	1	15	1
8	8	1	16	1	20	1
9	12	1	16	1	23	1
10	12	1	18	1	30	1
11	12	1	21	1	30	1
12	16	1	24	1	30	1
13	16	1	24	1	35	1
14	16	1	26	1	38	1
15	20	1	29	1	45	1
16	20	1	32	1	45	1
17	20	1	32	1	45	1
18	24	1	34	1	50	1
19	24	1	37	1	53	1

Solution: $x_1 = 0, x_2 = 0, x_3 = 3$. Maximum = 39.
Start with $x_1 = x_2 = x_3 = 0$. $\delta_3(16) = 1$ and so we add one to x_3. We have used up 5 units of the knapsack. There are 11 units left. $\delta_3(11) = 1$ and so we add one to x_3. We use up another 5 units and so we are left with 5. $\delta_3(6) = 1$. We add one more to x_3. There are now 1 units in the knapsack. $\delta_3(1) = 0$ and so we move to column 2. $\delta_2(1) = 0$ and so we move to column 1. $\delta(1) = 0$ and we are done.

Q2 A county chairwoman of a certain political party is making plans for an upcoming presidential election. She has received the services of 10 volunteer workers for precinct work and wants to assign them to five precincts in such a way as to maximize their effectiveness. She feels that it would be inefficient to assign a worker to more than one precinct, but she is willing to assign no workers to any one of the precincts if they can accomplish more in other precincts.
The following table gives the estimated increase in the number of votes for the party's candidate in each precinct if it were allocated the various number of workers.

Number	Precinct				
of Workers	1	2	3	4	5
0	0	0	0	0	0
1	4	7	5	6	4
2	10	11	10	11	12
3	15	16	15	14	15
4	18	18	18	16	17
5	22	20	21	17	20
6	24	21	22	18	22
7	26	25	24	23	22
8	28	27	27	25	25
9	32	25	30	28	28
10	33	28	34	32	30

Use dynamic programming to find all solutions to the problem of maximising votes.
Solution Let $f_r(w)$ be the maximum increase in votes in precincts $1, 2, \ldots, r$ assuming that w workers are assigned to them.

$$f_r(w) = \max_{0 \leq i \leq w} \{a_{i,r} + f_{r-1}(w - i)\}, \qquad r \geq 1,$$

where $a_{i,r}$ is the increased number of votes gained from i workers in precinct r.

$$f_0(w) = 0.$$

w	f_1	x_1	f_2	x_2	f_3	x_3	f_4	x_4	f_5	x_5
0	0	0	0	0	0	0	0	0		
1	4	1	7	1	7	0	7	0		
2	10	2	11	1,2	13	1	13	1		
3	15	3	17	1	17	0,2	18	1,2		
4	18	4	22	1	22	0,1,3	23	1,2		
5	22	5	26	2	27	1,2	28	1,2		
6	24	6	31	3	32	2,3	33	1,2		
7	26	7	34	3	37	3	38	1,2		
8	28	8	38	3	41	2,3	43	1,2		
9	32	9	40	3,4	46	3	48	2		
10	33	10	42	3,4,5	49	3,4	52	1,2	55	2

Solutions: Maximum = 55.

$$\begin{array}{lllll} x_1 = 3 & x_2 = 1 & x_3 = 3 & x_4 = 1 & x_5 = 2 \\ x_1 = 3 & x_2 = 1 & x_3 = 2 & x_4 = 2 & x_5 = 2 \\ x_1 = 2 & x_2 = 1 & x_3 = 3 & x_4 = 2 & x_5 = 2 \end{array}$$

Q3 We are given $2n$ sets $D_1, D_2, \dots D_n$ and $R_1, R_2, \dots, R_n$ where n is even. Also, $|D_i| + |R_i| = m$ for $i = 1, 2, \dots, n$. Find an algorithm that will check to see if the following is possible: Find a set $I \subseteq [n]$, $|I| = n/2$ such that

$$\sum_{i \in I} |D_i| \geq \sum_{i \in I} |R_i| \text{ and } \sum_{i \notin I} |D_i| \geq \sum_{i \notin I} |R_i|.$$

Your algorithm should run in time polynomial in m, n.

Solution: For $I \subseteq [n]$ let $D_I = \sum_{i \in I} |D_i|$ and $R_I = \sum_{i \in I} |R_i|$. Then let

$$f_{k,\ell}(a, b, c, d) = \begin{cases} 1 & \exists I \subseteq [k] : |I| = \ell \text{ and } D_I = a, D_{[k]\setminus I} = b, R_I = c, R_{[k]\setminus I} = d \\ 0 & otherwise \end{cases}$$

Then we have the recurrence

$$f_{k+1,\ell}(a, b, c, d) =$$
$$\begin{cases} 1 & f_{k,\ell-1}(a - |D_{k+1}|, b, c - |R_{k+1}|, d) + f_{k,\ell}(a, b - |D_{k+1}|, c, d - |R_{k+1}|) \geq 1 \\ 0 & otherwise \end{cases}$$

Having computed $f_{n,n/2}$ we can check to see whether or not there exist $a \geq c, b \geq d$ such that $f_{n,n/2}(a, b, c, d) = 1$. This takes $O((mn)^4 n^2)$ time, since this is the number of function evaluations we need to compute. Note that $D_{[n]} + R_{[n]} = mn$. (We can save a bit of time by only evaluating $f_{k,\ell}$ when $a + b + c + d = km$.)