9\28\12

All pairs shortest path problem.

Given a matrix $D = \| d_{ij} \|$ where

d_{ij} = length of edge (i,j).

walk

Find a shortest path between every pair of vertices.

Assume there are no negative cycles

Floyd's Algorithm

For $k = 1$ to n do

  For $i = 1$ to n do

    For $j = 1$ to n do

      $D_{ij} := \min\{ D_{i,j} , D_{i,k} + D_{k,j} \}$

    od

  od

od

$O(n^3)$

Claim

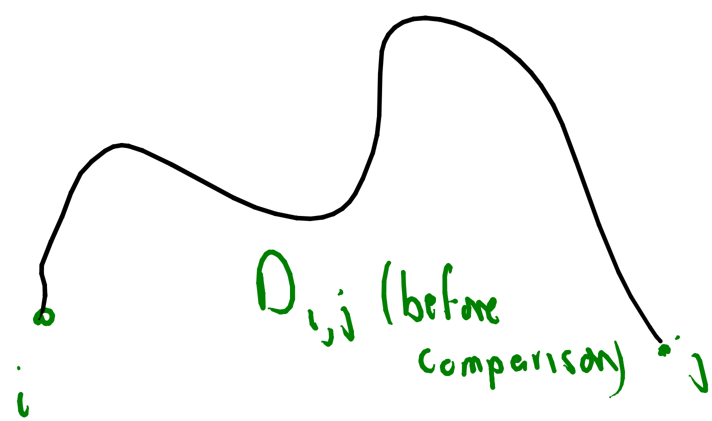
At the end of round k,

D_{ij} = min. length of walk from i to j

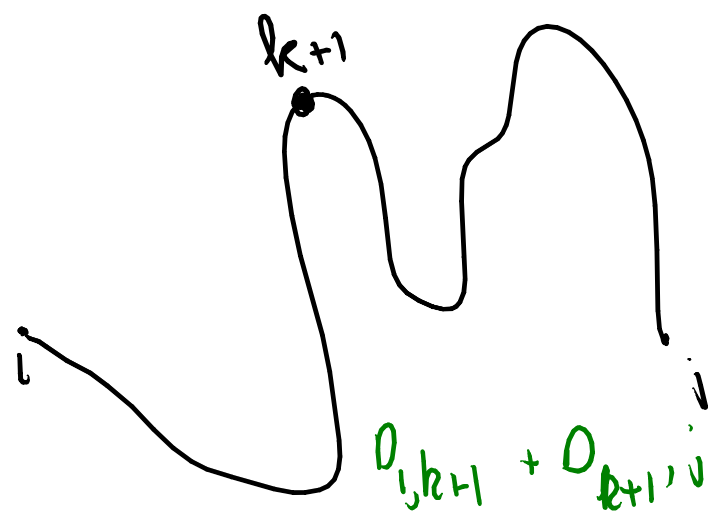
Proof by induction on k:

True for $k=0$

True for k: only uses internal vertices from $1, 2, \ldots, k, k+1$



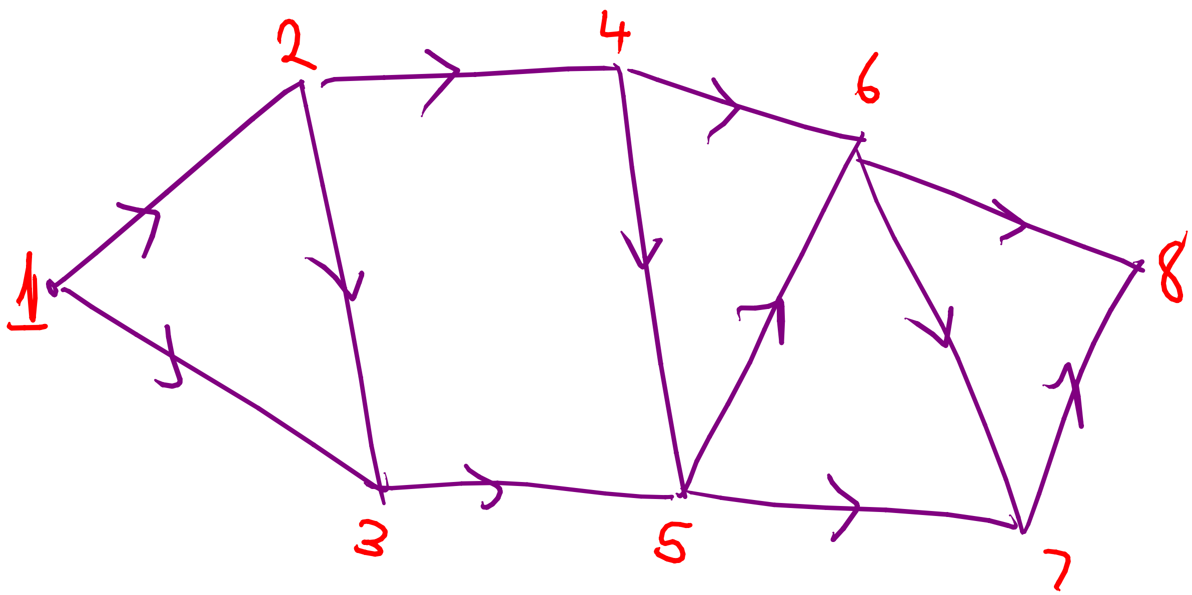
Don't use $k+1$ as an internal vertex



Digraphs without (oriented) cycles: DAG's

Directed Acyclic Graphs

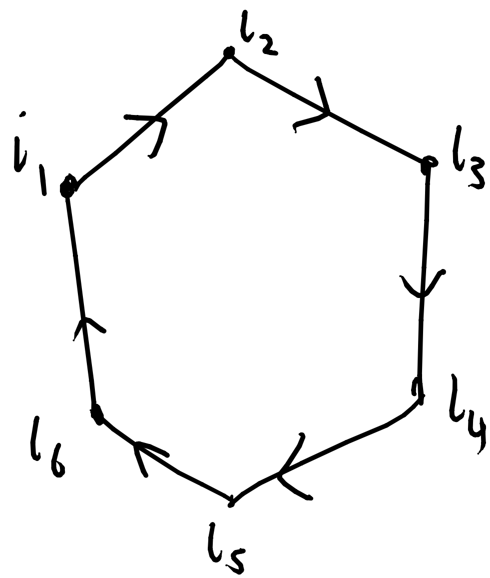
Linear (time) in # of arcs.



Topological ordering: # vertices 1, 2, .., n
s.t. every arc $i \longrightarrow j$ has $i < j$

Claim: a digraph has a topological ordering iff it has no oriented cycles.

$\longrightarrow$ only if



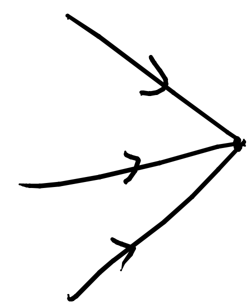
$i_1 < i_2 < i_3 < \dots < i_6$

Contradiction

$\longleftarrow$

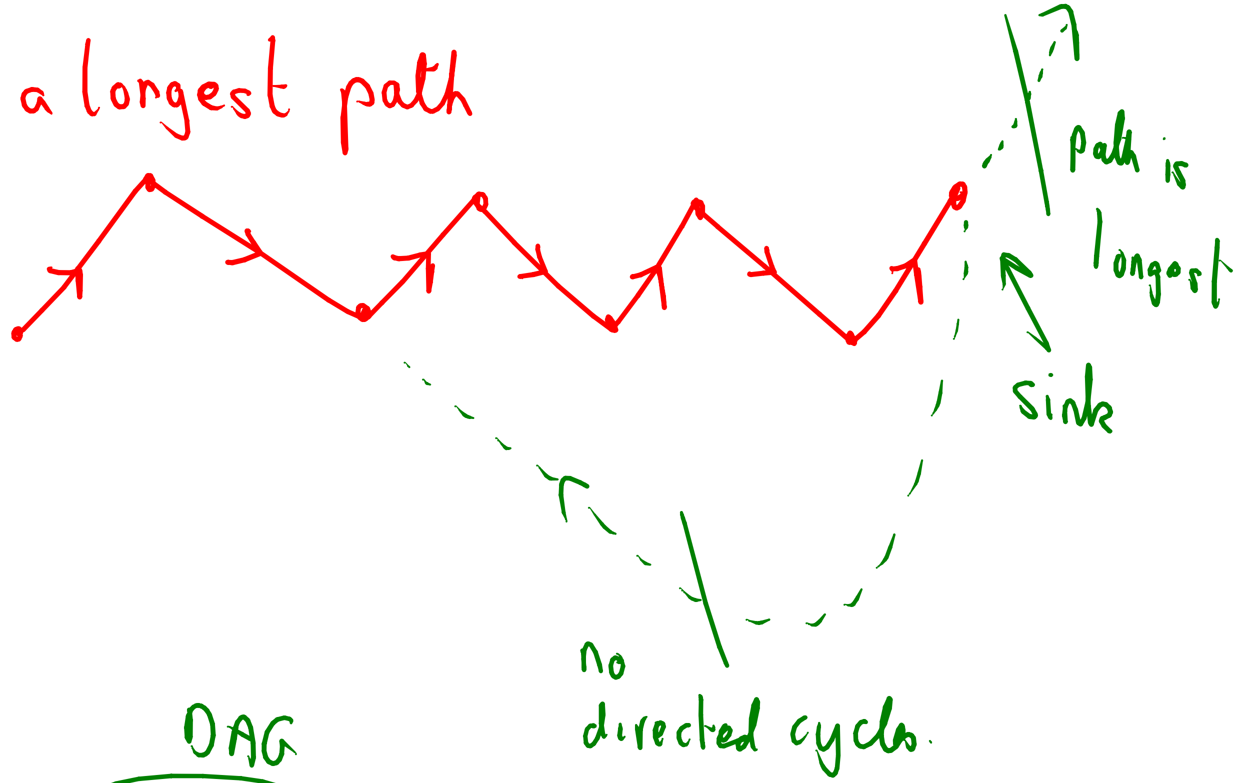
DAG $\Rightarrow$ $\exists$ ordering

DAG $\Rightarrow$ $\exists$ a "sink"

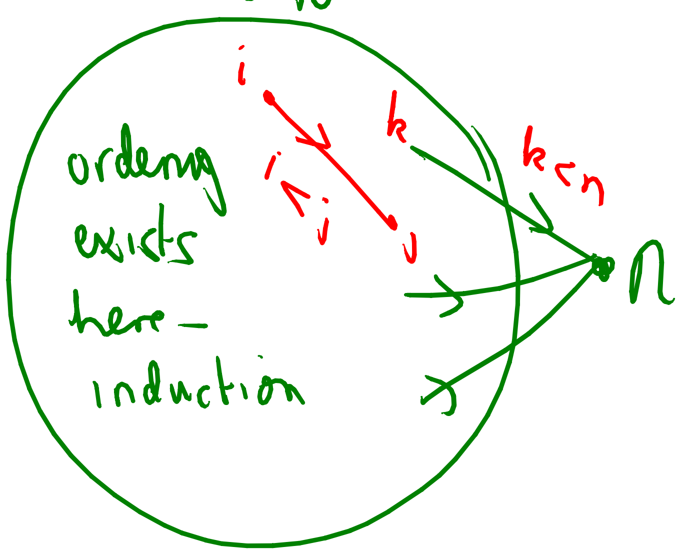


NO ARCS OUT

Take a longest path



DAG

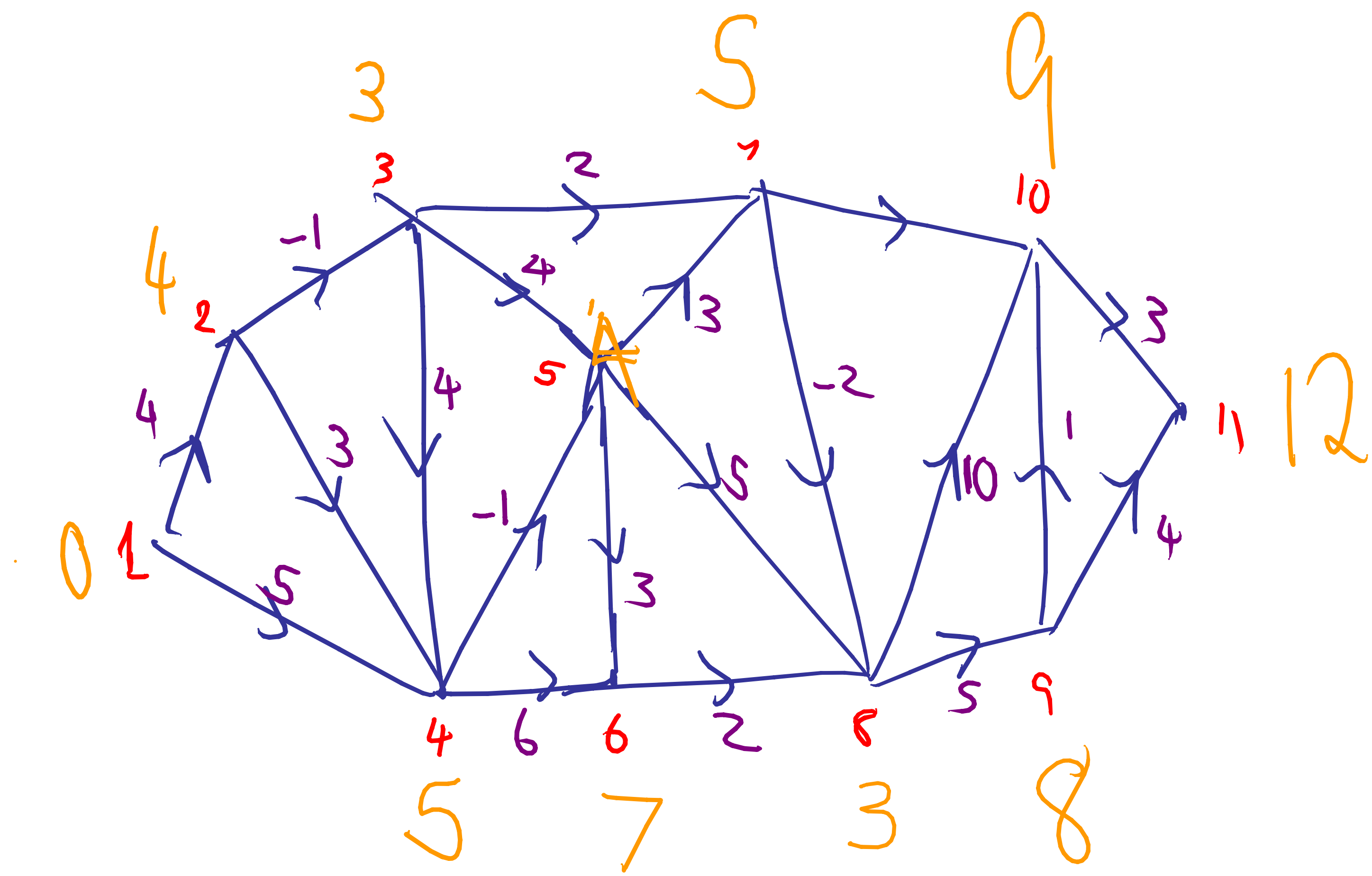


Suppose now that we want to find a shortest path from 1 to every other vertex.

d_j = min. path length to j

$$d_j = \min_{i<j} d_i + l_{ij}$$

$d_1 = 0$ and then I can compute $d_2, d_3, \ldots, d_n$



Solve

$d_j = \min d_i + l_{ij}$ $\xrightarrow{\text{DAG}}$ get a triangular system

$\oplus \quad \otimes$

$d_j = \bigoplus l_{ij} \otimes d_i$