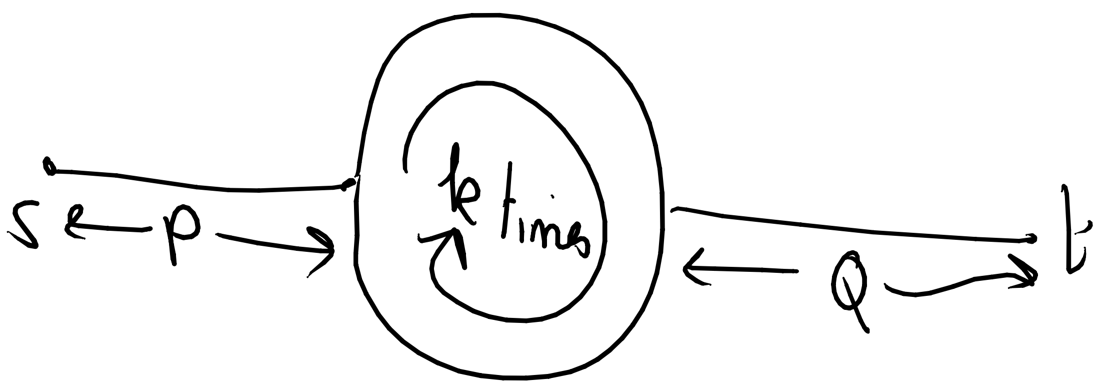
9/21/12

Shortest paths with negative arc lengths.

Problem in general is NP-hard

We replace problem by find shortest walk to every vertex.

First problem: Suppose $\exists$ cycle C such that $\ell(C) = \lambda < 0$.

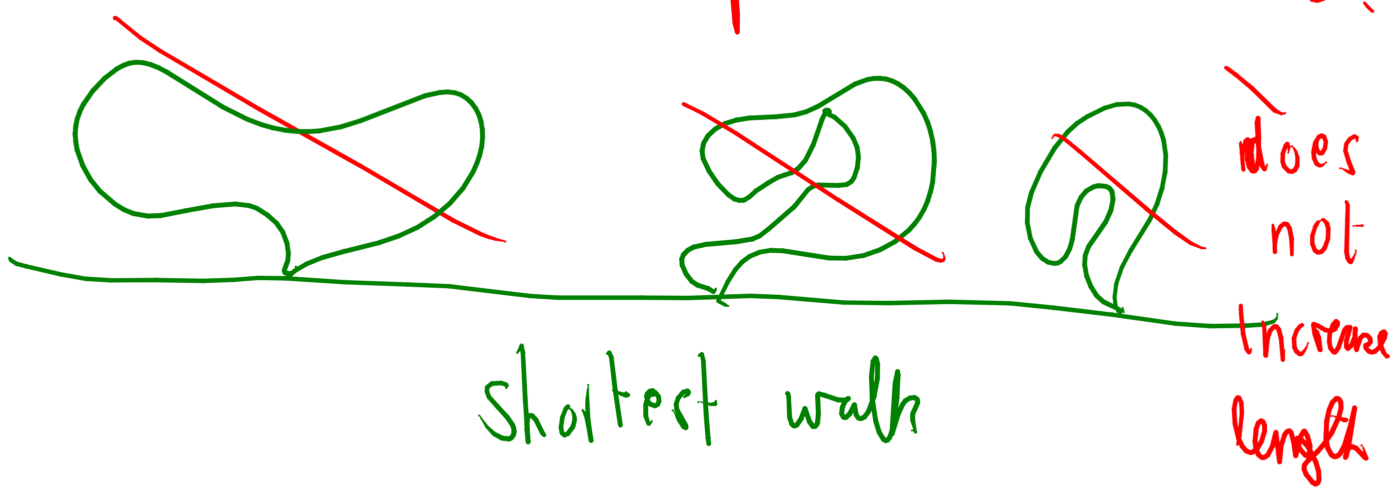


$$\text{length} = \ell(P) + k\lambda + \ell(Q) + \ell(\tfrac{1}{2}C).$$

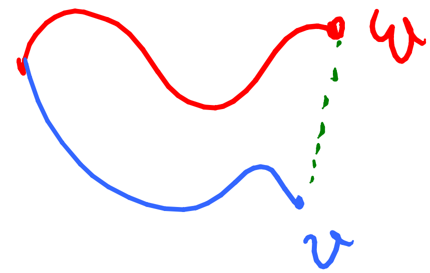
$$\longrightarrow -\infty \quad \text{with } k$$

So, we are only going tackle shortest walk problem where $\ell(C) \geq 0$ for all directed cycles C.

A shortest walk from s to t gives us a shortest path from s to t.



Optimality condition



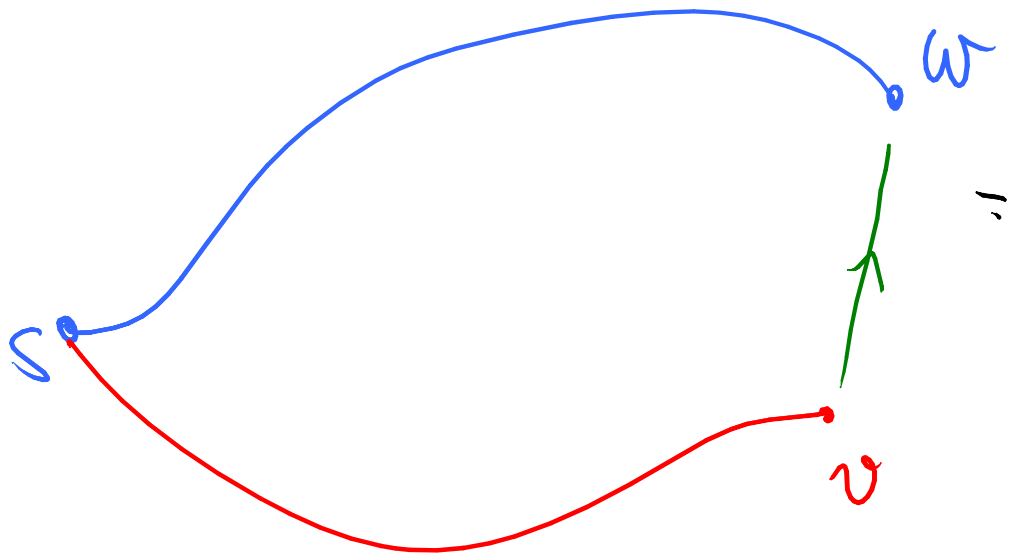
Suppose that for all $v \in V$ we have a walk W_v of length $d(v)$ from s to v.

Claim: These walks are all shortest iff

$$(*) \qquad d(w) \le d(v) + \ell(v,w), \quad \forall (v,w) \in E$$

Proof

Suppose $d(w) > d(v) + \ell(v,w)$



Red + Green < Blue

Suppose that ⊛ holds and

W is any walk from s to w.

We need to show that $\ell(W) \geq d(w)$.

Suppose $W = (s = w_0, w_1, w_2, \ldots, w_k = w)$

$$
\begin{aligned}
l(W) = \;& l(w_0, w_1) \geq d(w_1) - d(w_0) \\
+\;& l(w_1, w_2) \geq d(w_2) - d(w_1) \\
& \hphantom{l(w_1, w_2) \geq\;} + \\
+\;& l(w_2, w_3) \geq d(w_3) - d(w_2) \\
& \hphantom{l(w_1, w_2) \geq\;} + \\
+\;& \\
& \vdots \\
+\;& \\
& l(w_{k-1}, w_k) \geq d(w_k) - d(w_{k-1}) \\
& \hphantom{l(w_1, w_2) \geq\;} +
\end{aligned}
$$

$$
l(W) \geq d(\underset{\uparrow \atop w}{w_k}) - \underbrace{d(w_0)}_{0}
$$

Algorithm

Start with some walks W_v, $v \in V$

$d(v) = \ell(W_v)$

repeat:

If $\exists (v,w)$ such that

$$d(w) > d(v) + \ell(v,w)$$

then replace W_w by $W_v + (v,w)$

until: optimality condition

Finite termination

$\sum_v d(v)$ strictly decreases each step.

Never get same set of walks twice.

Assuming we always remove cycles, set of walks will be a set of paths and there is a finite no. of sets of paths

Simple algorithm

$E = \{e_1, e_2, \ldots, e_m\}$ = edges of D

$e_i = (x_i, y_i)$

ROUND

repeat

for $i = 1$ to m do

if $d(y_i) > d(x_i) + l(x_i)$ then

$d(y_i) \leftarrow d(x_i) + l(x_i)$

$W_{y_i} \leftarrow W_{x_i} + (x_i, y_i)$ od

until optimal [i.e. no change]

Claim

Running time = $O(mn)$

After k rounds, $d(v)$ is correct for every v for which there is a shortest walk with $\leq k$ edges.

Induction: $k=0$ ✓

Inductive step