9\17\12

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \qquad \pi = (3, 3, 4, 4)$$

Optimality condition:

$$y_i = \min_j \left[c_{ij} + \alpha y_j \right]$$

Algorithm

Check for optimality of current π

If $\exists\, i$ such that

$$y_i > c_{ik} + \alpha y_k = \min_j \left[c_{ij} + \alpha y_j \right]$$

then make $\pi'(i) = k$.

new π

$\pi \rightarrow$ check

$\uparrow$

new

(evaluate)

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \qquad \overline{\pi} = (3, 3, 4, 4)$$

$$y_1 = 5 + \frac{1}{2} y_3 = 13/2$$

$$y_2 = 2 + \frac{1}{2} y_3 = 7/2$$

$$y_3 = 2 + \frac{1}{2} y_4 = 3$$

$$y_4 = 1 + \frac{1}{2} y_4 = 2$$

Check Optimality

$i = 1$

$6 + \frac{1}{2} \times \frac{13}{2}$

$4 + \frac{1}{2} \times \frac{7}{2}$ *

$5 + \frac{1}{2} \times 3$

$7 + \frac{1}{2} \times 2$

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3, 3, 4, 4)$$

Check Optimality

$i = 2$

$3 + \frac{1}{2} \times \frac{13}{2}$

$9 + \frac{1}{2} \times \frac{7}{2}$

$2 + \frac{1}{2} \times 3 \quad *$

$5 + \frac{1}{2} \times 2$

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3, 3, 4, 4)$$

Check Optimality

$i = 3$

$4 + \frac{1}{2} \times \frac{13}{2}$

$3 + \frac{1}{2} \times \frac{7}{2}$

$7 + \frac{1}{2} \times 3$

$2 + \frac{1}{2} \times 2$ *

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3, 3, 4, 4)$$

Check Optimality

$i = 4$

$5 + \frac{1}{2} \times \frac{13}{2}$

$3 + \frac{1}{2} \times \frac{7}{2}$

$8 + \frac{1}{2} \times 3$

$1 + \frac{1}{2} \times 2$ *

New $\pi = (2, 3, 4, 4)$

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3, 3, 4, 4)$$

Evaulate new π

$$y_1 = 4 + \tfrac{1}{2} y_2 = 23/4$$

$$y_2 = 2 + \tfrac{1}{2} y_3 = 7/2$$

$$y_3 = 2 + \tfrac{1}{2} y_4 = 3$$

$$y_4 = 1 + \tfrac{1}{2} y_4 = 2$$

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \qquad \pi = (3, 3, 4, 4)$$

Check Optimality

$i = 1$

$6 + \frac{1}{2} \times \frac{23}{4}$

$4 + \frac{1}{2} \times \frac{7}{2}$ *

$5 + \frac{1}{2} \times 3$

$7 + \frac{1}{2} \times 2$

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3, 3, 4, 4)$$

Check Optimality

$i = 2$

$$3 + \frac{1}{2} \times \frac{23}{4}$$

$$9 + \frac{1}{2} \times \frac{7}{2}$$

$$2 + \frac{1}{2} \times 3 \quad *$$

$$5 + \frac{1}{2} \times 2$$

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3, 3, 4, 4)$$

Check Optimality

$i = 3$

$$4 + \frac{1}{2} \times \frac{23}{4}$$

$$3 + \frac{1}{2} \times \frac{7}{2}$$

$$7 + \frac{1}{2} \times 3$$

$$2 + \frac{1}{2} \times 2$$

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3, 3, 4, 4)$$

Check Optimality

$i = 4$

$$5 + \frac{1}{2} \times \frac{23}{4}$$

$$3 + \frac{1}{2} \times \frac{7}{2}$$

$$8 + \frac{1}{2} \times 3$$

$$1 + \frac{1}{2} \times 2$$

NO CHANGE

$\Downarrow$

$(2,3,4,4)$ is optimal.

$$\begin{bmatrix} 6 & 4 & 5 & 7 \\ 3 & 9 & 2 & 5 \\ 4 & 3 & 7 & 2 \\ 5 & 3 & 8 & 1 \end{bmatrix} \quad \pi = (3,3,4,4)$$

Verifying optimality conditions [O.C.]

Suppose π & y satisfy O.C.

Take any other policy $\hat{\pi}$ and $\hat{y}$

$$\hat{y}_i = c_{i\hat{\pi}(i)} + \alpha \hat{y}_{\hat{\pi}(i)}$$

$$y_i \leq c_{i,\hat{\pi}(i)} + \alpha y_{\hat{\pi}(i)}$$

$$\hat{y}_i - y_i \geq \alpha\left(\hat{y}_{\hat{\pi}(i)} - y_{\hat{\pi}(i)}\right) \geq \alpha^2\left(\hat{y}_{\hat{\pi}^2(i)} - y_{\hat{\pi}^2(i)}\right)$$

So, repeating,

$$\hat{y}_i - y_i \geq \alpha^t \left[\hat{y}_{\hat{\pi}^t(1)} - y_{\hat{\pi}^t(1)} \right] \forall t$$

$$\Downarrow$$

$$\hat{y}_i - y_i \geq 0$$

Suppose O.C. doesn't hold.

Go from $\Pi \longrightarrow \hat{\Pi}$ (new Π)

$$\hat{y}_i = c_{i,\hat{\pi}(i)} + \alpha \hat{y}_{\hat{\pi}(i)}$$

$$y_i \geq c_{i,\hat{\pi}(i)} + \alpha y_{\hat{\pi}(i)} \quad \& > \text{ for at least one } i$$

by choice of $\hat{\Pi}$

$\hat{y}_i - y_i < \alpha(\quad)$

$\leq \alpha^2(\quad)$

$\vdots$

$\Rightarrow \hat{y}_i \leq y_i, \forall i$ and $\hat{y}_i < y_i$

Problem can be solved by linear programming.

Maximize $y_1 + \cdots + y_n$

s.t.:

$$y_i \leq c_{ij} + \alpha y_j$$

Claim: in an optimum solution to

$$y_i = \min_j \left[c_{ij} + \alpha y_j \right]$$

$$= c_{i, \pi(i)} + \alpha y_{\pi(i)}$$