9\7\11

Knapsack Problem

Maximize $$p_1 x_1 + p_2 x_2 + \cdots + p_n x_m$$

subject to $$w_1 x_1 + w_2 x_2 + \cdots + w_n x_n \leq W$$

$$x_1, x_2 \ \cdots x_n \geq 0 \text{ \& integer}$$

Assumptions are: $w_1, \ldots, w_n$ & $p_1, \ldots, p_n$ & $W > 0$.

Why knapsack problem?

Scout is going on a trip

Item j has "usefulness" p_j
and weight w_j per unit.

Scout can only carry W in weight.

Wants to maximise usefulness.
Has to take whole items.

Without "wholeness" requirement. One constraint
LP.
Choose j to maximise $\frac{p_j}{w_j}$ and fill with this.

D.P. approach.

Solution = sequence of decisions.

Decision n = how many of item n to put in.

Suppose I decide to put in x_n of item n; similar problem

$$n \longrightarrow n-1; \quad W \longrightarrow W - w_n x_n$$

So let:

How long to compute $f_n(W)$?

$O(nW^2)$

$f_r(w)$ = maximum usefulness obtainable if I can only use items $1, 2, \cdots r$ and there is w left in knapsack.

$$= \max_{0 \le x_r \le \frac{w}{w_r}} \left[p_r x_r + f_{r-1}(w - w_r x_r) \right]$$

$f_0(w) = 0$ OR $f_1(w) = \left\lfloor \frac{w}{w_1} \right\rfloor p_1$

$$f_r(w) = \max \begin{cases} f_{r-1}(w) & x_r = 0 \\ p_r + f_r(w - w_r) & x_r \geq 1 \end{cases}$$

Tempo di esecuzione

$$O(nW).$$

Example: $n = 4$, $\underline{w} = (1, 2, 3, 4)$ $\quad W = 11$

$\underline{p} = (1, 3, 6, 10)$

w	f_1	b_1	f_2	b_2	f_3	b_3	f_4	b_4
0	0	0	0	0	0	0	0	0
1	1	1	1	0	1	0	1	0
2	2	1	3	1	3	0	3	0
3	3	1	4	1	6	1	6	0
4	4	1	6	1	7	1	10	1
5	5	1	7	1	9	1	11	1
6	6	1	9	1	12	1	13	1
7	7	1	10	1	13	1	16	1
8	8	1	12	1	15	1	20	1
9	9	1	13	1	18	1	21	1
10	10	1	15	1	19	1	23	1
11	11	1	16	1	21	1	26	1