Note Title 9/27/2011

9/28/11

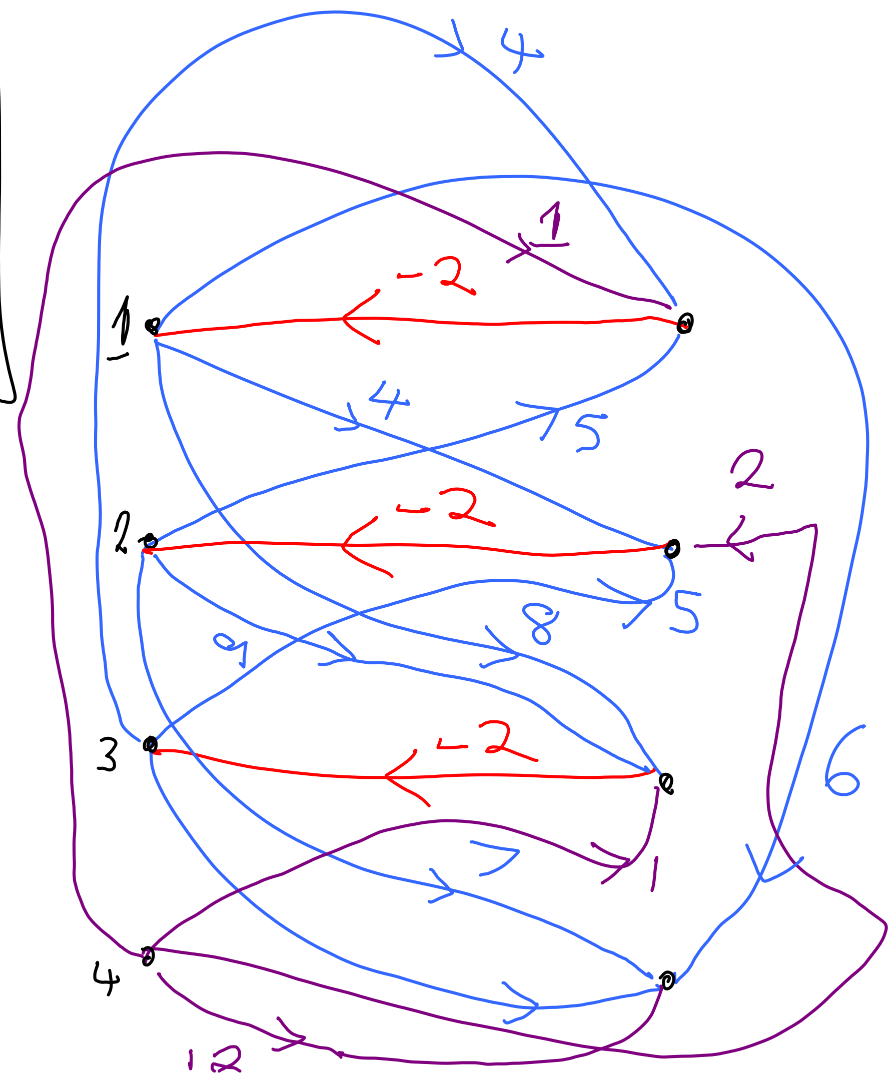
Assignment Problem:

Given solution to $k \times k$ problem we solve a shortest path problem to solve the $(k+1) \times (k+1)$ problem.

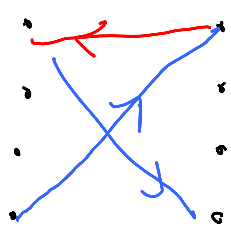
$$C = \begin{bmatrix} 2 & 4 & 8 & 6 \\ 5 & 2 & 9 & 7 \\ 4 & 5 & 2 & 8 \\ 1 & 2 & 1 & 12 \end{bmatrix}$$

= is 4x4 optimum

0 solve 3x3 problem.



Find shortest path from ~~4~~1L ⟶ 4R.

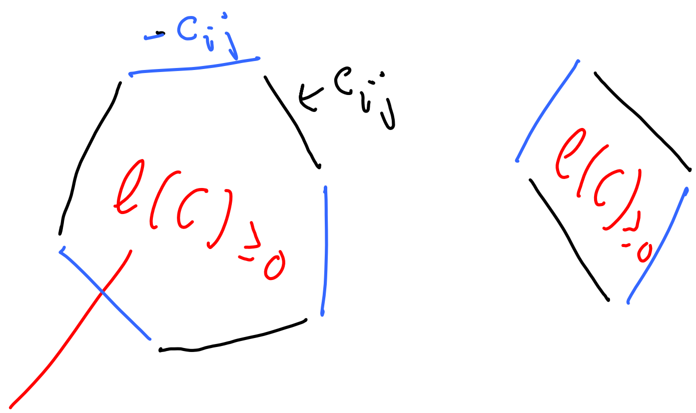


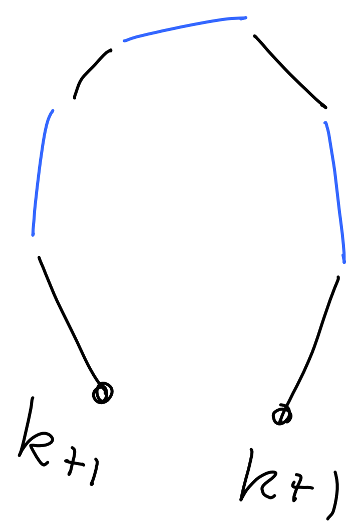
??

Suppose M_k is a minimum cost matching for $k \times k$ problem.

Suppose M is an matching for $(k+1) \times (k+1)$ problem

$M_k \oplus M =$





edges in at most one of M_k, M

because M_k is min. $k \times k$ matching

Linear Program

ALP

Minimise $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to $\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, \ldots, n$

$\sum_{j=1}^{n} x_{ij} = 1, \qquad i = 1, 2, \ldots, n$

$x_{ij} = 1$ iff $(i,j) \in M$

$\cancel{x_{ij} = 0 \text{ or } 1}$

$x_{ij} \geq 0$

Constraint Matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
& & & & 1 & 1 & 1 & 1 & & & & & & & & \\
& & & & & & & & 1 & 1 & 1 & 1 & & & & \\
& & & & & & & & & & & & 1 & 1 & 1 & 1 \\
1 & & & & 1 & & & & 1 & & & & 1 & & & \\
& 1 & & & & 1 & & & & 1 & & & & 1 & & \\
& & 1 & & & & 1 & & & & 1 & & & & 1 & \\
& & & 1 & & & & 1 & & & & 1 & & & & 1
\end{bmatrix}
$$

Claim

Every basic solution to the constraints in ALP is integral.

$A\underset{\sim}{x} = b$

$$A = \begin{array}{c} \begin{array}{ccc} \underset{\sim}{x}_B & & \underset{\sim}{x}_N \end{array} \\ [\, B \,\vdots\, N \,] \end{array}$$

re-arrange columns

$\boxed{\text{Basic Solution to LP}}$

$A\underset{\sim}{x} = \underset{\sim}{b} \equiv Bx_B + Nx_N = b$

B is non-singular

$x_N = 0$

$x_B = B^{-1}\underset{\sim}{b}$

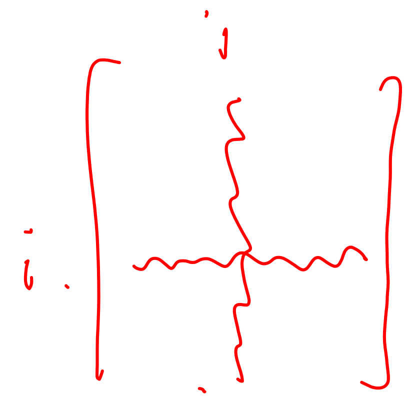
Proof

The matrix of constraints is

totally unimodular"

i.e. all submatrices have determinant $0, \pm 1$

$$B^{-1} = \frac{\operatorname{adj} B}{\det(B)}; \qquad \operatorname{adj} B = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & \\ & & & \\ & & & B_{n,m} \end{bmatrix}$$



$B_{i,j} = (-1)^{i+j} \det \text{Rest}$

All matrices of the following form have determinant $0, \pm 1$.

$k \times k$ matrix M

every sub-determinant is $0, \pm 1$

$$
\begin{array}{cccccc}
\multicolumn{3}{c}{\text{2 non-zero's}} & \multicolumn{3}{c}{\text{one non-zero}} \\
\end{array}
$$

$$
\left[\begin{array}{cccccc}
 & 1 & & & & \\
1 & & & & & \\
 & & & 1 & & -1 \\
 & -1 & & & & \\
\hline
 & & 1 & & & \\
 & & & & -1 & \\
1 & & -1 & & &
\end{array}\right]
$$

Induction on k.

Base Case $k =$

General Case:

a) $\exists$ column in M with one non-zero

b) Every column has 2 non-zeros

$\det M = \det M' \rightarrow \begin{bmatrix} 1 & \\ & \boxed{M'} \end{bmatrix}$

b) Every column of M has 2 non-zeros

$$\left[\begin{array}{cccc} & 1 & & 1 \\ 1 & & & \\ & -1 & 1 & \\ \hline & & & 1 \\ 1 & & 1 & \end{array}\right]$$

Sum of rows above line

=

Sum of rows below line

$\Downarrow$

det M = 0.