9\23\11

Shortest paths with negative arc lengths, but no negative cycles i.e. $\ell(C) \geq 0$ for all directed cycles C.

Optimality Condition

Suppose that we want to find a shortest walk from s to every vertex.

Suppose that for all $v \in V$ we have a walk W_v of length $d(v)$ from s to v.

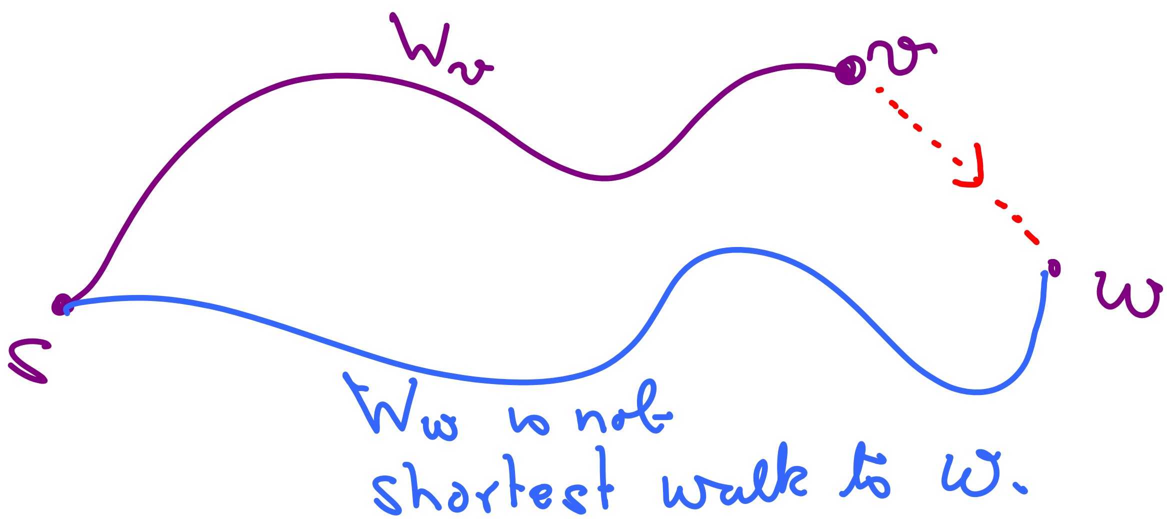
Claim: These walks constitute a set of shortest walks iff

$$(*) \quad d(w) \leq d(v) + \ell(v,w), \quad \forall v,w$$

(1)

Suppose $d(w) > d(v) + \ell(v,w)$

then $\ell(W_w) > \ell(W_v, v)$



(II) Suppose ⊛ holds and W is any walk from s to v.

We need to show that $l(W) \geq d(v)$.

Suppose $W = (s = w_0, w_1, w_2, \ldots, w_k = v)$

$d(w_i) + l(w_i, w_{i+1}) \geq d(w_{i+1})$

$$
\left.
\begin{aligned}
l(W) = {} & l(w_0, w_1) && \geq d(w_1) - d(w_0) \\
+ {} & l(w_1, w_2) && \geq d(w_2) - d(w_1) \\
+ {} & l(w_2, w_3) && \geq d(w_3) - d(w_2) \\
& \quad \vdots && \quad \vdots \\
+ {} & l(w_i, w_{i+1}) && \geq d(w_{i+1}) - d(w_i) \\
& \quad \vdots && \quad \vdots \\
+ {} & l(w_{k-1}, v) && \geq d(v) - d(w_{k-1})
\end{aligned}
\right\}
\begin{aligned}
& \text{Sum} = \\
& d(v) - \\
& d(w_0) \\
& = d(v)
\end{aligned}
$$

Algorithm

Start with some walks $W_v, v \in V$

$d(v) = l(W_v)$ for all $v \in V$

repeat

If $\exists (v, w)$ such that $d(w) > d(v) + l(v, w)$

then replace W_w by (W_v, w)

until optimality condition holds

Finite termination:

(i) $\sum_v d(v)$ strictly decreases by at least one, assuming arc lengths are integers.

(ii)

$$\sum_v d(v) \geq n \sum_e \min\{0, l(e)\}.$$

because no walk will use same edge twice.

(iii) On termination, optimality condition holds.

Good idea to go through all edges in one pass.

Algorithm

$E = \{e_1, e_2, \dots, e_m\}$ = edges of D

$e_i = (x_i, y_i)$

ROUND
repeat
    for $i = 1, 2, \dots, m$
        if $d(y_i) > d(x_i) + l(e_i)$ then
            $d(y_i) \leftarrow d(x_i) + l(e_i)$
            $W(y_i) \leftarrow (W(x_i), y_i)$
until optimality

n = # vertices: # rounds $\leq n-1$

After k rounds $d(x)$ is correct $\forall x$ such that $\exists$ shortest path $s \Rightarrow x$ using $\leq k$ edges — Induction