9/21/11

Shortest Path Problem

Digraph $D = (V, A)$

Length $l : A \to \mathbb{R}$ [edge a has length $l(a)$]

If P is a path the

$$l(P) = \sum_{a \in P} l(a)$$

Single Source Problem

Given source $s \in V$, find a shortest path from s to all $v \in V$.

Case 1

Non-negative arc lengths.

Assume $l(a) \geq 0, \; \forall a \in A$

Dijkstra's Algorithm

Initially we have $d(s) = 0$ and $d(v) = \infty$ for all $v \neq s$

[$d(v)$ is our current estimate of the shortest distance from s to v.]

$S = \emptyset$ — $S = \{v :$ $d(v)$ is known to be correct

$x = s$

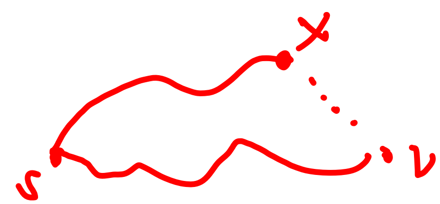
repeat

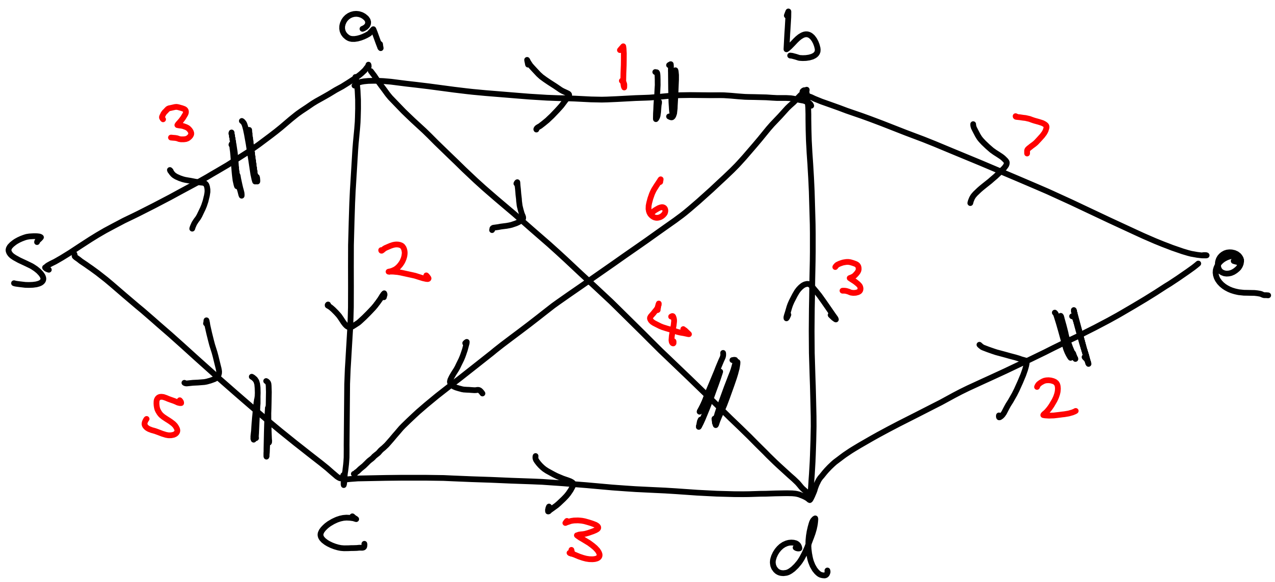
$\{$ Process x : $\forall v \notin S$; $d(v) \leftarrow \min\big(d(v), d(x) + \ell(x,v)\big)$

$S \leftarrow S \cup \{x\}$

$d(y) = \min\{v \notin S : d(v)\}$

$x \leftarrow y$





	s	a	b	c	d	e
d	$\boxed{0}$	∞	∞	∞	∞	∞
d	$\boxed{0}$	$\boxed{3}$	∞	5	∞	∞
d	$\boxed{0}$	$\boxed{3}$	$\boxed{4}$	5	7	11
d	$\boxed{0}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	7	11
d	$\boxed{0}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$	$\boxed{7}$	9

Generalisation

Time dependent arc lengths

$$\ell : A \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

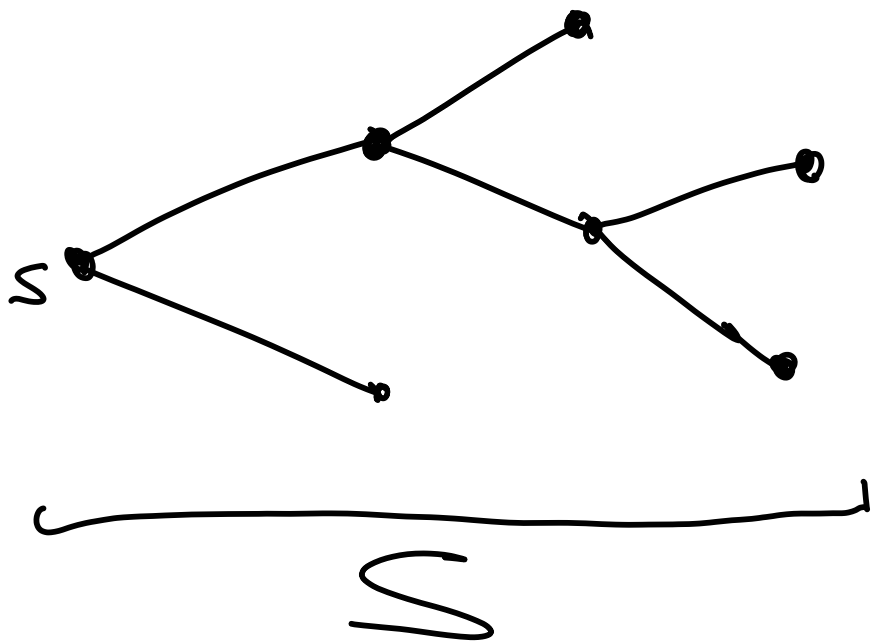
$\ell(a,t)$ = length of $a = (x,y)$ if arrive at x at time t.

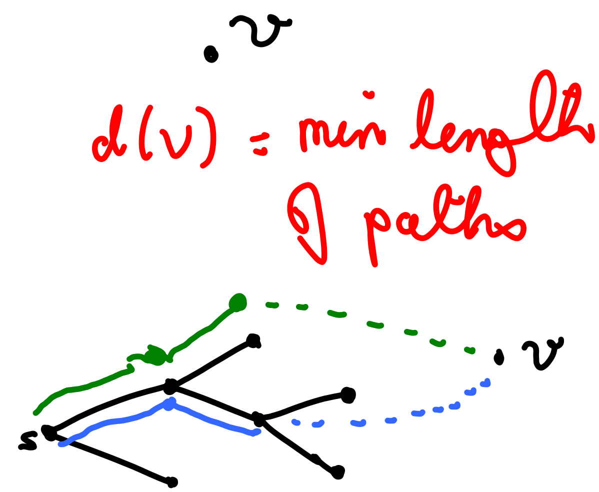
$$P = (u_0, u_1, u_2, \ldots, u_k)$$

$$\ell(P) = \underbrace{\ell((u_0,u_1), 0)}_{\xi_1} + \underbrace{\ell((u_1,u_2), \xi_1)}_{\xi_2} +$$

$$\ell((u_2,u_3), \xi_1+\xi_2) + \cdots$$

Dijkstra's algorithm can be adapted to this situation





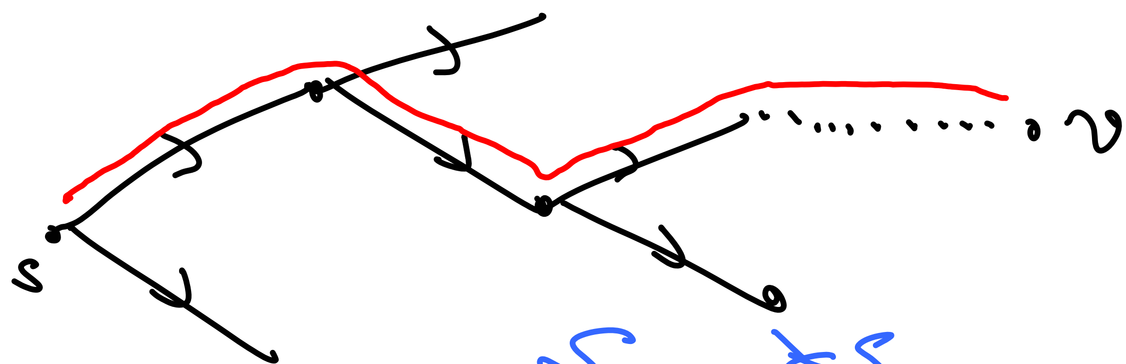
Claim: $d(v)$ is correct for $v \in S$.

Proof: By induction on $|S|$.

True for $|S| = 1$.

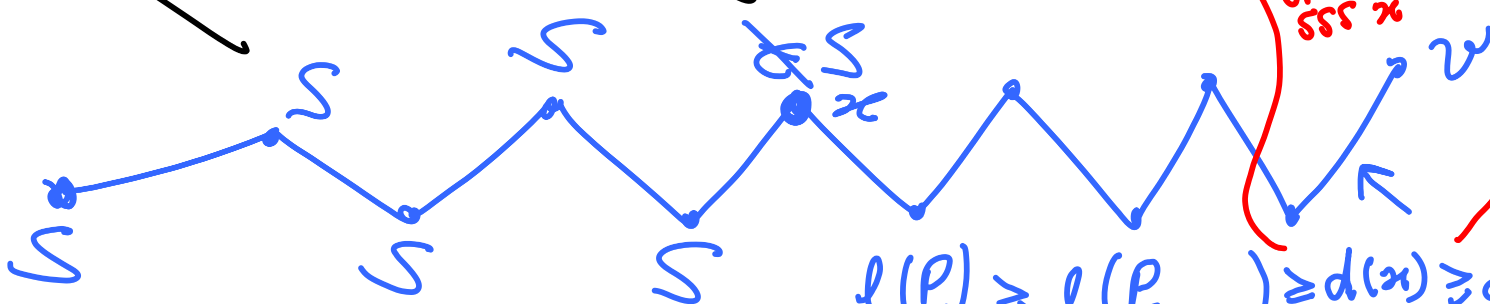
Consider adding vertex to S.

v minimizes d



v was closest

$d(x)$ = shortest path to x of form $sss\,x$.



$\ell(P) \geq \ell(P_{s \to x}) \geq d(x) \geq d(v)$

Case 2: allow arc lengths to be negative.

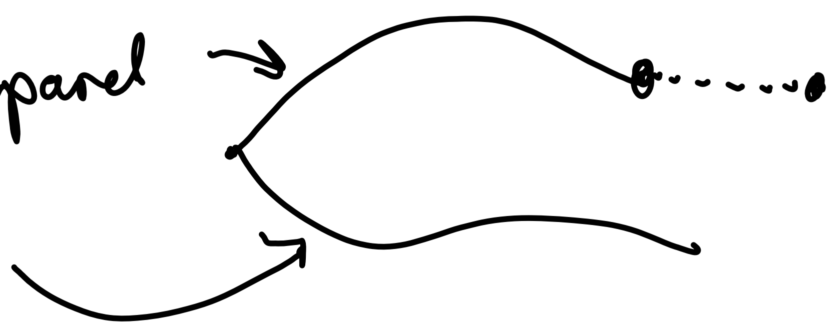
Our algorithm for $\ell \geq 0$ found a shortest walk from s to each vertex

Because $\ell \geq 0$, shortest walks and shortest paths coincide

Computationally, it is "easy" to find shortest walks.

We compared

with



Really finding shortest walks

While # paths $a \to b$ is finite
($\Rightarrow \exists$ shortest path)

walks $a \to b$ is ∞

If there exists cycle with $\ell(C) < 0$
then shortest walks may not exist.
On the other hand if $\ell(C) \geq 0, \forall C$
then shortest walk = shortest path.