10\7\11

Primal-Dual Assignment Algorithm [Max]

Find $\underline{y} = [u_1, \dots u_n, v_1, \dots, v_n]$ and $\underline{x} = (x_{ij})$ s.t.

(i) $\underline{x}$ is feasible

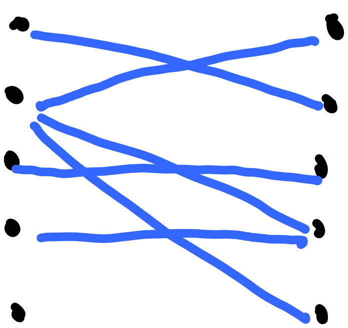
(ii) $\underline{y}$ is dual feasible

(iii) $(u_i + v_j - c_{ij})\, x_{ij} = 0, \ \forall i, j$ Complementary Slackness

$$
\begin{array}{c|ccccc}
 & 6 & 7 & 5 & 5 & 6 \\
\hline
0 & 3 & 7 & 4 & 4 & 3 \\
0 & 6 & 4 & 3 & 5 & 6 \\
0 & 4 & 1 & 5 & 3 & 2 \\
0 & 5 & 3 & 4 & 5 & 3 \\
0 & 4 & 2 & 3 & 1 & 4
\end{array}
$$

0

$u_i + v_j = c_{ij}$

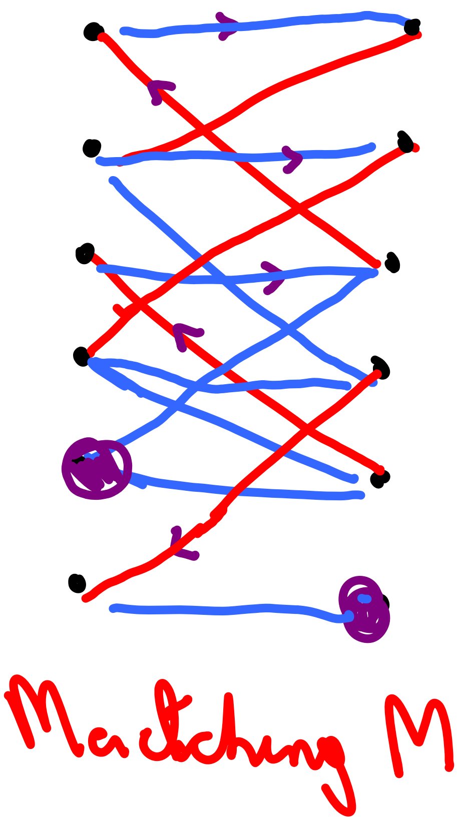


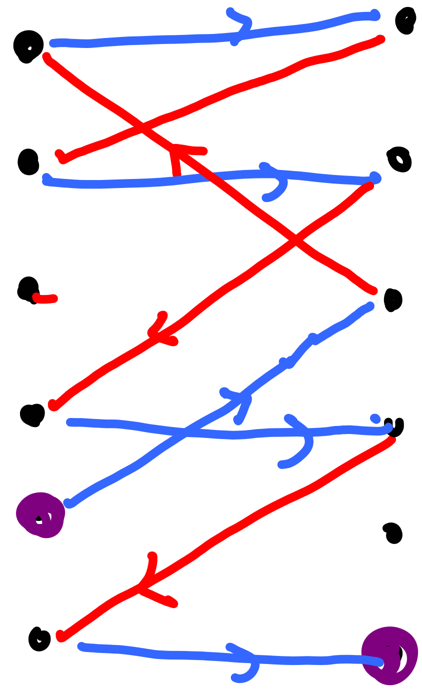
Start by choosing some dual feasible $\underline{y}$.

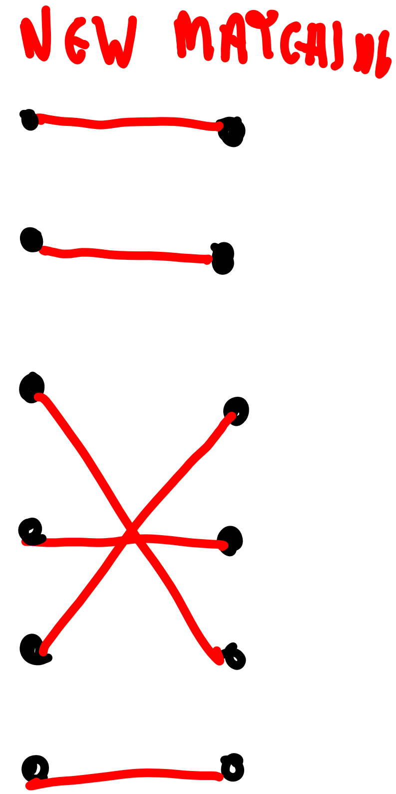
$u_i = 0, \quad i = 1, 2, \dots, n$

$v_j = \max_i c_{ij}$

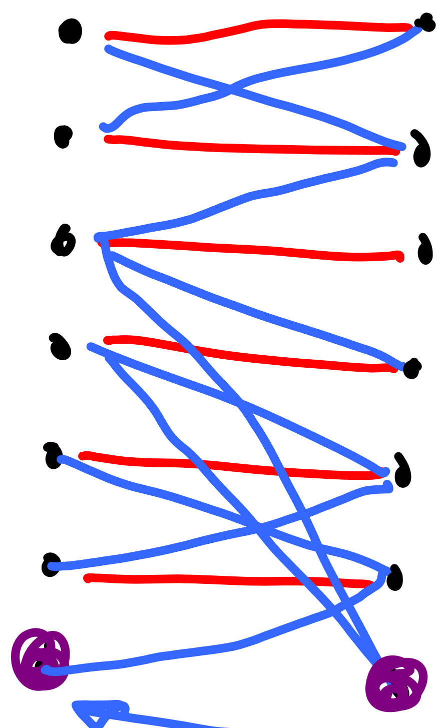
$G_=$ has edge (i, j) iff $u_i + v_j = c_{ij}$



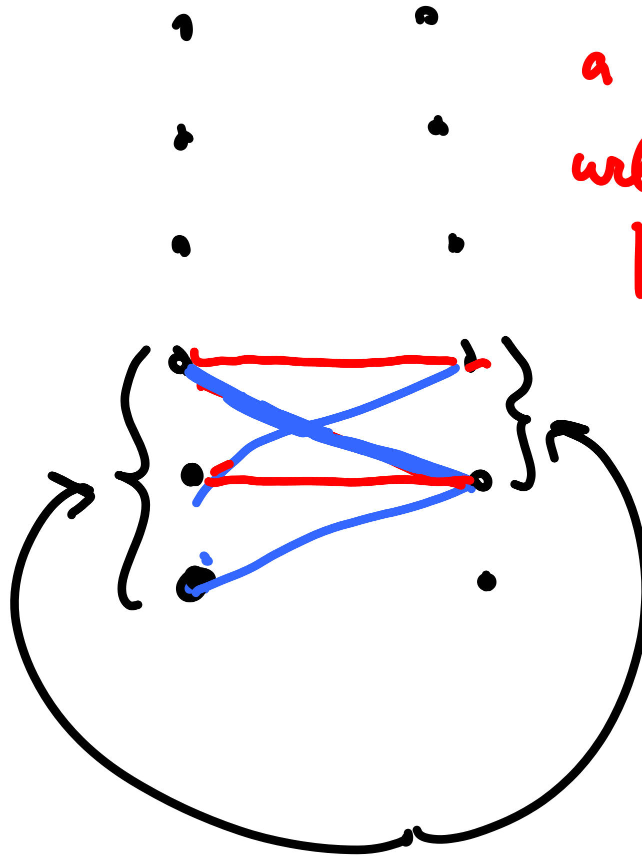




Find an alternating path from ● to ● — e.g. use B.F.S.

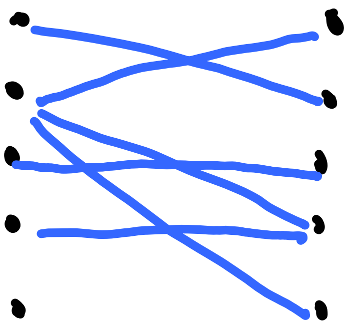


Do BFS from



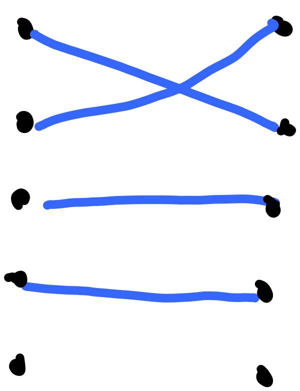
We find a set S with $|N(S)| < |S|$ as promised by Hall's thm.

Vertices reachable from • by an A.P.



Use augmenting paths to find maximum matching

(a) Pick some matching

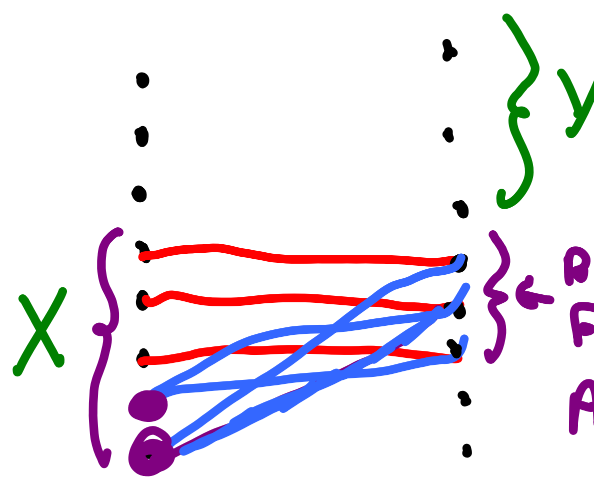


Find a maximum matching M in $G_=$

(a) $|M| = n$. Done

$x_{ij} > 0 \Rightarrow u_i + v_j = c_{ij}$

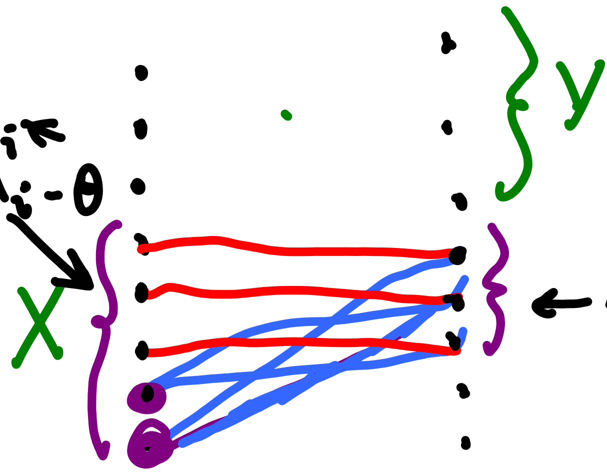
(b) $|M| < n$



NO EDGES $G_=$ $X \to Y$ also can reach a vertex of y

REACHABLE FROM ● by A.P.'s

(b) $|M| < n$



NO EDGES $G_=$ $X \rightarrow Y$ also can reach a vertex of Y

$u_i \leftarrow u_i - \theta$

$\leftarrow v_j \leftarrow v_j + \theta$

$i \in X$ & $j \in Y$ then $c_{ij} > u_i + v_j$

$$\theta = \min_{\substack{i \in X \\ j \in Y}} c_{ij} - u_i - v_j \,, \quad \theta > 0$$

(i) All edges used to construct X, Y stay in $G_=$.

$(u_i - \theta + v_j + \theta)$

(ii) We gain an $X:Y$ edge.

After $\leq n^2$ changes in dual we find an augmenting path

$O(n^4)$ algorithm