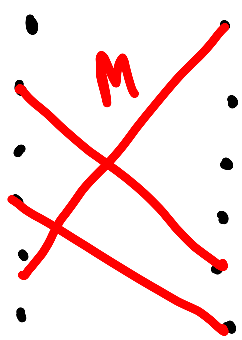
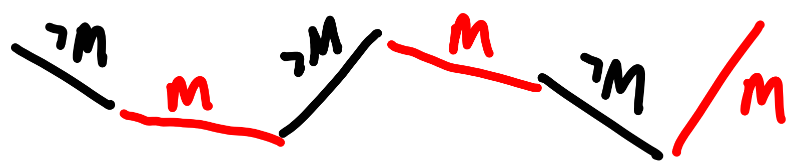
10\5\11

Finding a maximum size matching in a bipartite graph.

Alternating Paths

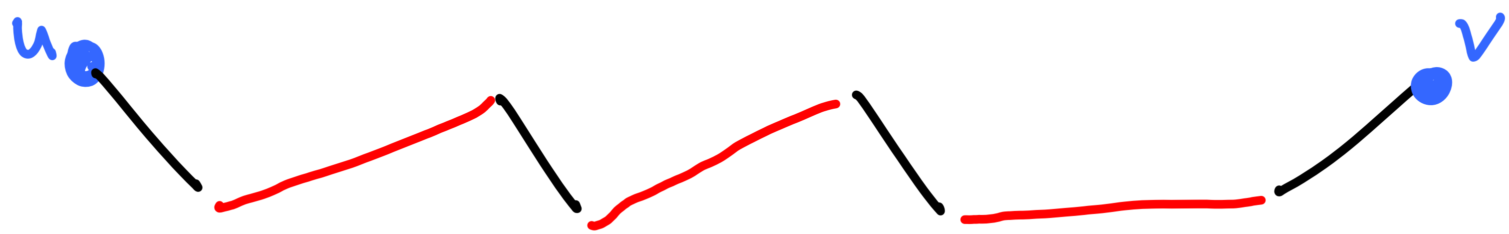


An alternating path:



Augmenting path:

an alternating path that starts and ends at vertices not covered by M.



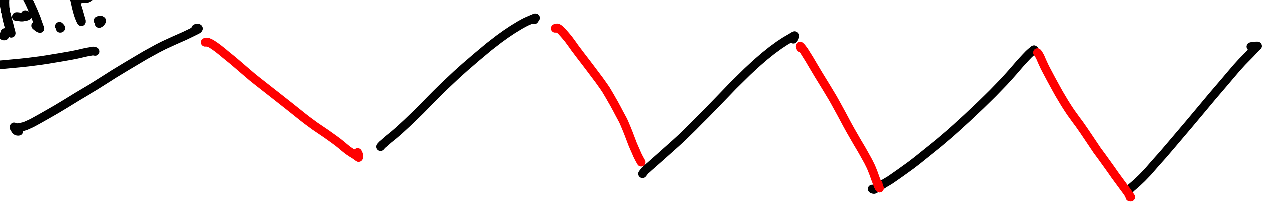
No edge of M contains u or v.

Claim

M is of maximum size iff $\nexists$ an augmenting path.

Proof

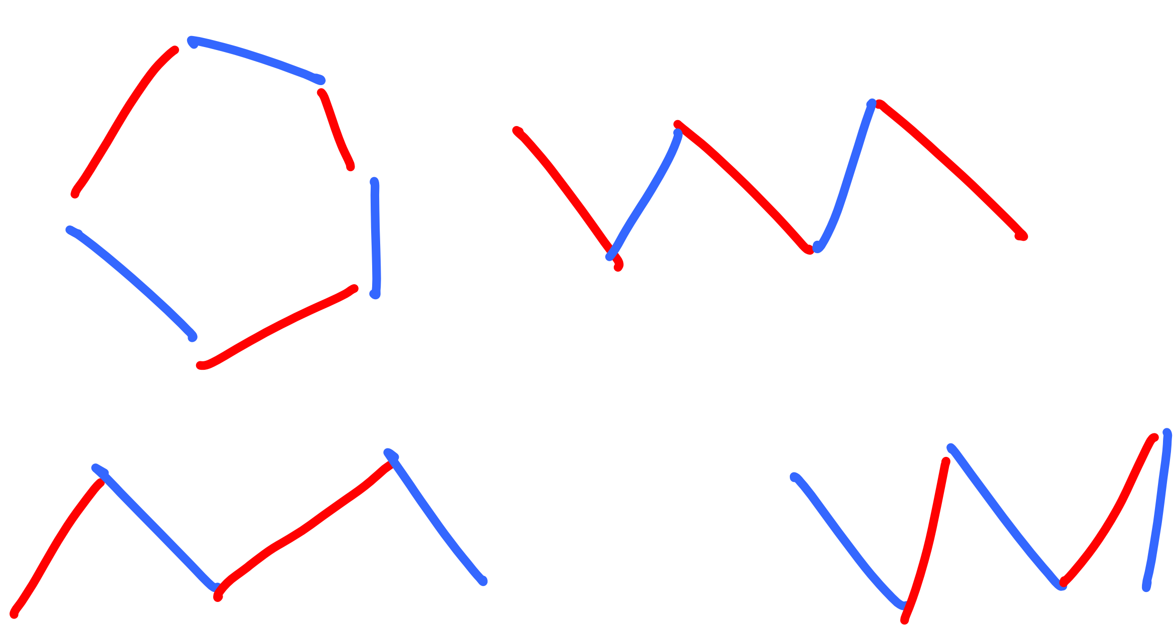
(1) $\exists$ A.P.



Replace by and get a larger matching

(ii) Suppose $|M'| > |M|$

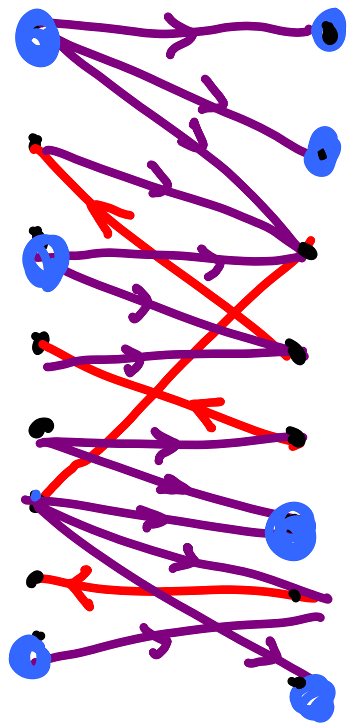
$M' \oplus M =$



Edges in $M' \cap M$ are missing from picture

Must get at least one augmenting.

Given M it is easy to search for an augmenting path.



Search for a path from ● on the left to a ● on the right

Primal-Dual Algorithm for the assignment problem.

Maximise $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}$; $\underline{c_{ij} \geq 0}$

s.t.

$$\sum_{i=1}^{n} x_{ij} = 1, \ \forall j$$

$$\sum_{j=1}^{n} x_{ij} = 1, \ \forall i$$

$$x_{ij} \geq 0$$

<u>Dual</u>

Min: $\sum_{i=1}^{n} u_i + \sum_{j=1}^{n} v_j$

$$u_i + v_j \geq c_{ij}$$

Linear Program

$$\max.\ \underline{c}^T \underline{x}$$

$$\text{s.t.}\quad A\underline{x} \le \underline{b}$$

$$\underline{x} \ge 0$$

$$\Downarrow$$

$$A\underline{x} + \underline{s} = \underline{b}$$

$$\underline{s}, \underline{x} \ge 0$$

Dual Linear Program

$$\min.\ \underline{b}^T \underline{u}$$

$$\text{s.t.}\quad A^T\underline{u} \ge \underline{c}$$

$$\underline{u} \ge 0$$

$$\Downarrow$$

$$A^T\underline{u} - \underline{t} = c$$

$$\underline{u}, \underline{t} \ge 0$$

$$\underline{b}^T \underline{u} - \underline{x}^T \underline{c} =$$

$$(\underline{x}^T A + \underline{s}^T)\underline{u} - \underline{x}^T(A^T \underline{u} - \underline{t})$$

$$= \underline{s}^T \underline{u} + \underline{x}^T \underline{t}$$

$\underline{u}$, $\underline{x}$ are both optimal iff

$s_i u_i = 0, \ \forall i$

$x_j t_j = 0, \ \forall j$

Optimality: Complementary slackness.

$$x_{ij}(u_i + v_j - c_{ij}) = 0.$$

Algorithm

(a) Choose feasible $\underline{u}, \underline{v}$

(b) Let $G_=$ = graph induced by edge (i,j) s.t.

$$u_i + v_j = c_{ij}$$

Find a maximum matching M in $G_=$

(c) $|M| = n$; finished. $|M| < n$ alter $\underline{u}, \underline{v}$