10\26\11

$$\begin{array}{c|c} A \backslash B & v \\ \hline u & \left[\begin{array}{c} \\ \text{PAY}(u,v) \\ \\ \end{array}\right] \end{array}$$

Game is stable iff

$$\max_u \text{ROWMIN}(u) = \min_v \text{COLMAX}(v)$$

Question: how to play the game when it isn't stable

We use randomization.

A chooses $p_1, p_2, \dots, p_m$ and chooses u with probability p_u

B chooses $q_1, \dots, q_n$

$$\begin{array}{cc} & \underline{q} \\ \underline{p} & \begin{bmatrix} \\ PAY(\underline{p}, \underline{q}) \\ \\ \end{bmatrix} \end{array}$$

$$PAY(\underline{p}, \underline{q}) = E(\text{winnings for } A)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

Can we always find $\underline{p}, \underline{q}$ so that this game is stable?

We compute P_A, P_B and see if $P_A = P_B$

$$P_A = \max_{\underline{p}} \min_{\underline{q}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

$$= \max_{\underline{p}} \min_{\underline{q}} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} p_i \right) q_j$$

$$= \max_{\underline{p}} \underline{\min_{\underline{q}} \sum_{j=1}^{n} c_j(\underline{p})\, q_j} \quad ; \qquad c_j(\underline{p}) = \sum_{i=1}^{m} a_{ij} p_i$$

$$= \max_{\underline{p}} \min_{j=1,\dots,n} c_j(\underline{p})$$

$$P_A = \max_{\underline{p}} \underbrace{\min_{j=1,\dots,n} c_j(\underline{p})}_{\xi}$$

$$= \text{maximum } \xi$$

$$\xi \leq \sum_{i=1}^{m} a_{ij} p_i \qquad j = 1, 2, \dots, n$$

$$p_1 + p_2 + \dots + p_m = 1$$

$$p_1, p_2, \dots, p_m \geq 0$$

$$P_B = \min_{\underline{q}} \max_{\underline{p}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, p_i\, q_j$$

$$= \min_{q} \max_{p} \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}\, q_j \right) p_i$$

$$= \min_{\underline{q}} \max_{\underline{p}} \sum_{i=1}^{m} r_i(q)\, p_i \; ; \qquad r_i(\underline{q}) = \sum_{j=1}^{n} a_{ij}\, q_j$$

$$= \min_{\underline{q}} \min_{i=1,2,\dots,m} r_i(q)$$

$$P_B = \min_{\underline{q}} \underbrace{\max_{i=1,2,\dots,m} r_i(\underline{q})}_{\eta}$$

$$= \text{minimum } \eta$$

$$\eta \geq \sum_{j=1}^{n} a_{ij} q_j \qquad i = 1, 2, \dots, m$$

$$q_1 + q_2 + \dots + q_n = 1$$

$$q_1, \dots, q_n \geq 0$$

maximum ξ

$$\xi \leq \sum_{i=1}^{m} a_{ij} p_i \qquad j = 1, 2, \dots, n$$

$$p_1 + p_2 + \dots + p_m = 1$$

$$p_1, p_2, \dots, p_m \geq 0$$

DUAL PAIR

minimize η

$$\eta \geq \sum_{j=1}^{n} a_{ij} q_j \qquad i = 1, 2, \dots, m$$

$$q_1 + q_2 + \dots + q_n = 1$$

$$q_1, \dots, q_n \geq 0$$