10\24\11

Two person Zero-Sum Games

Matrix P

Player B

$$\text{Player } A \quad \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & \cdots & j & \cdots & n \end{array} \\ \left[\begin{array}{ccc} & & \\ & & \\ & a_{ij} & \\ & & \\ & & \end{array} \right] \end{array}$$

B pays a_{ij} to A

Game is played repeatedly.

A chooses a row i

B chooses a col. j

A writes i on a piece of paper.

B writes j on a piece of paper.

They put paper into a box. Neither sees choice until box is opened by C

Example

1.

$$\begin{array}{c c} & \begin{array}{ccc} R & P & S \end{array} \\ \begin{array}{c} R \\ P \\ S \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

No saddle point

2. Tennis

B receives - assume

A serves

$$\begin{array}{c c} & \begin{array}{ccc} W & M & B \end{array} \\ \begin{array}{c} \text{Wide} \\ \text{Middle} \\ \text{Body} \end{array} & \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \end{array}$$

No saddle point

$$\begin{bmatrix} 3 & 6 & 5 & 4 \\ 2 & 4 & 2 & 5 \\ 1 & 6 & 3 & 7 \\ -3 & 1 & 2 & 100 \end{bmatrix}$$

3 is minimum in row and maximum in column.

- saddle point.

One strategy for A: choose row that maximises the row min

Here row = 1

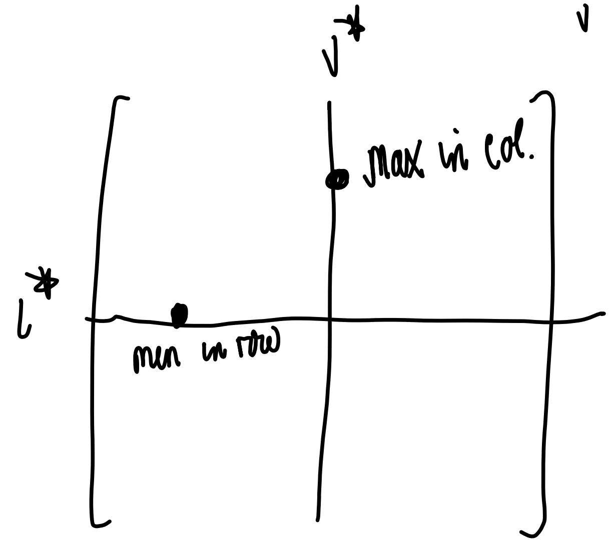
Strategy for B: choose column to minimise the column max.

Here col = 1

If no saddle point, what happens:

A chooses i^* when $\text{Rowmin}(i^*) = \max_i \text{Rowmin}(i)$

B chooses j^* where $\text{Colmax}(j^*) = \min_j \text{Colmax}(j)$



Change notation:

$$\begin{array}{cc} & v \\ u & \begin{bmatrix} \\ \\ \mathrm{PAY}(u,v) \\ \\ \\ \end{bmatrix} \end{array}$$

(u_0, v_0) is <u>stable</u> if

$$\mathrm{PAY}(u_0, v_0) = \mathrm{ROWMIN}(u_0)$$ B has no incentive to change

$$= \mathrm{COLMAX}(v_0)$$ A has no incentive to change.

$$P_A = \max_u \mathrm{ROWMIN}(u)$$ A can always get this

$$P_B = \min_v \mathrm{COLMAX}(v)$$ B can always pay at most this

Thm

(i) $P_A \leq P_B$

(ii) $P_A = P_B$ iff there is a stable pair.

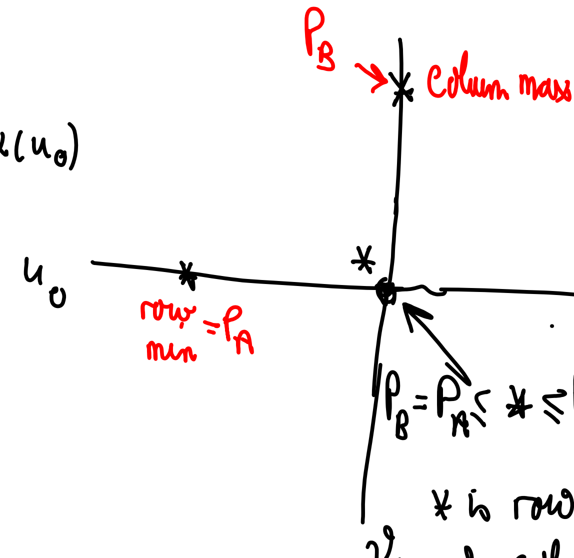
Proof

(i) $\mathrm{ROWMIN}(u) \leq \mathrm{PAY}(u,v) \leq \mathrm{COLMAX}(v)$

(ii)

$$P_A = P_B$$

P_A: ROWMIN(u_0) $\qquad$ P_B: COLMAX(u_0)



$P_B = P_A \leq * \leq P_B = P_A$

$*$ is row min & col max

$\Downarrow$

stability

Nash Equilibrium

(u_0, v_0) is stable

$$P_A \geq \mathrm{ROWMIN}(u_0) = \mathrm{PAY}(u_0, v_0) = \mathrm{COLMAX}(v_0) \geq P_B$$

$$\Rightarrow P_A = P_B$$