10\14\11

# Gomory Cuts:

Pure Integer Program:

$$\min x_0 = \underline{c}^T \underline{x}$$

$$A\underline{x} = \underline{b}$$

$$x_0, \underline{x} \geq 0 \quad \& \text{ integer.}$$

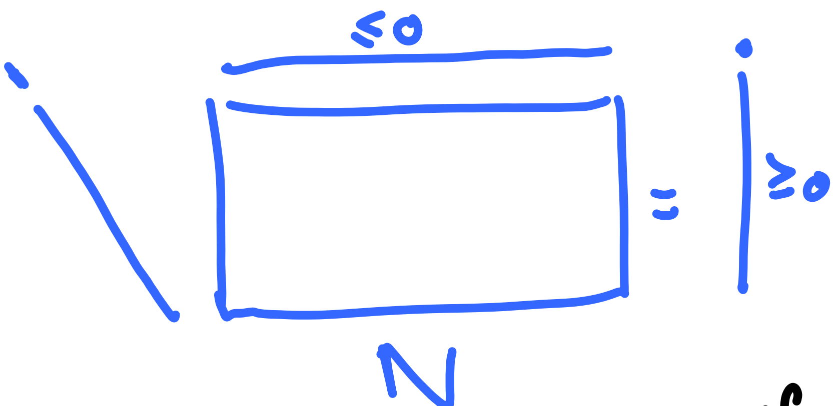
Suppose we solve the linear relaxation.

End up with a simplex tableau

$$x_i + \sum_{j \in N} b_{ij} = b_{i,0}$$

for all basic variable $x_i \in \{0\} \cup I$.

indices of the basic variables




If $b_{i,0}$ is integer for all $i$ then this soln IP as well.

We suppose not i.e. there exist $i$ such that $b_{i,0} \notin \mathbb{Z}$

$$x_i + \sum_{j \in N} b_{ij} x_j = b_{i0}$$

We will find a contraint that is satisfied by all non-negative integral solutions to but is not satisfied by putting

$$x_i = b_{i,0}\,,\quad x_j = 0,\ j \in N.$$

In general consider the equation

$$(*) \qquad a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \qquad a_1, \ldots, a_n, b \in \mathbb{R}.$$

Write $a_i = \lfloor a_i \rfloor + f_i$ where $0 \leq f_i < 1$

$6\frac{1}{2} = 6 + \frac{1}{2}$ and $-7\frac{1}{4} = -8 + \frac{3}{4}$

$$b = \lfloor b \rfloor + f$$

Substituting, we get

$$\sum_{j=1}^{n} f_j x_j - f = \lfloor b \rfloor - \sum_{j=1}^{n} \lfloor a_j \rfloor x_j$$

$$(**) \qquad \sum_{j=1}^{n} f_j x_j - f = \lfloor b \rfloor - \sum_{j=1}^{n} \lfloor a_j \rfloor x_j$$

This holds for all solutions to (*).

Suppose in addition that $x_1, x_2, \ldots, x_n \in \mathbb{Z}$ then RHS of (**) is an integer.

So $\displaystyle\sum_{j=1}^{n} f_j x_j - f$ is also an integer.

$$\sum_{j=1}^{n} f_j x_j - f \quad \text{is also an integer.}$$

Suppose now that $x_1, \ldots, x_n$ are also $\geq 0$.

Then

$$\sum_{j=1}^{n} f_j x_j - f \geq \cancel{-f} \geq 0 \quad \text{as } f < 1.$$

All non-negative integer solutions to $(*)$ satisfy

$$\sum_{j=1}^{n} f_j x_j \geq f$$

Apply this to

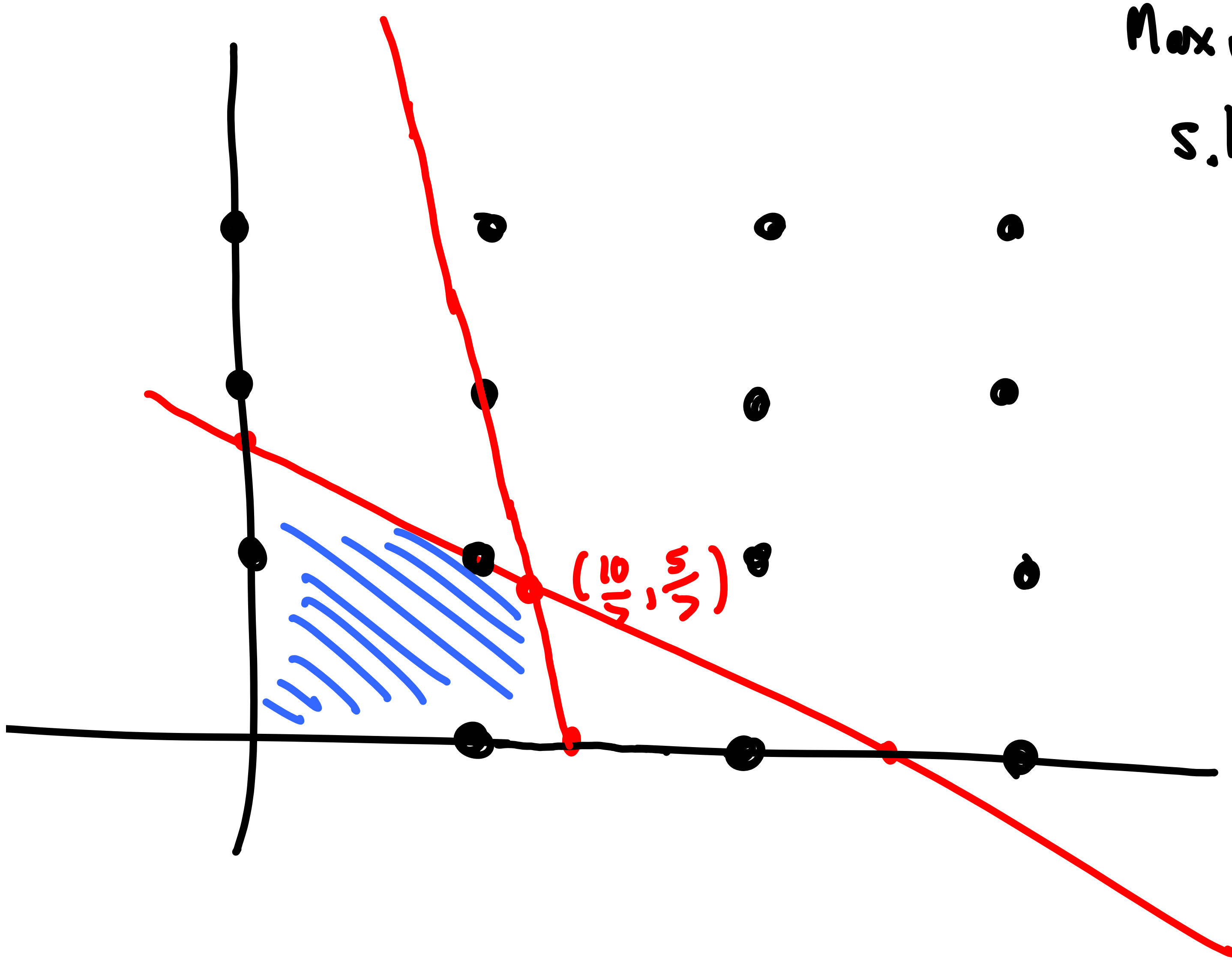
$$x_i + \sum_{j \in N} \bar{b}_{ij} x_j = \bar{b}_{i0}$$

$\underline{1} = 1 + 0$

$$0 x_i + \sum_{j \in N} f_j x_j \geq f$$

LP optimum does not satisfy this, as $x_j = 0$, $j \in N$ and $f > 0$.

So we have a <u>cut</u>




Maximise $3x_1 + 4x_2$

s.t.

$$2x_1 + 3x_2 \leq 5$$

$$4x_1 - x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

& integer.

| B.V. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | R |
|---|---|---|---|---|---|
| $x_0$ | $-3$ | $-4$ | | | 0 |
| $x_3$ | 2 | 3 | 1 | | 5 |
| $x_4$ | 4 | $-1$ | | 1 | 5 |

| B.V. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | R |
|---|---|---|---|---|---|
| $x_0$ | $-\frac{1}{3}$ | | $\frac{4}{3}$ | | $\frac{20}{3}$ |
| $x_2$ | $\frac{2}{3}$ | 1 | $\frac{1}{3}$ | | $\frac{5}{3}$ |
| $x_4$ | $\frac{14}{3}$ | | $\frac{1}{3}$ | 1 | $\frac{20}{3}$ |

DUAL FEASIBLE

LP optimum

| B.V. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\xi_1$ | R |
|---|---|---|---|---|---|---|
| $x_0$ | | | $\frac{19}{14}$ | $\frac{1}{14}$ | | $\frac{50}{7}$ |
| $x_2$ | | 1 | $\frac{2}{7}$ | $-\frac{1}{7}$ | | $\frac{5}{7}$ |
| $x_1$ | 1 | | $\frac{1}{14}$ | $\frac{3}{14}$ | | $\frac{10}{7}$ |
| $\xi_1$ | | | $-\frac{5}{14}$ | $\boxed{-\frac{1}{14}}$ | 1 | $-\frac{1}{7}$ |

integer variable

DUAL SIMPLEX NOW

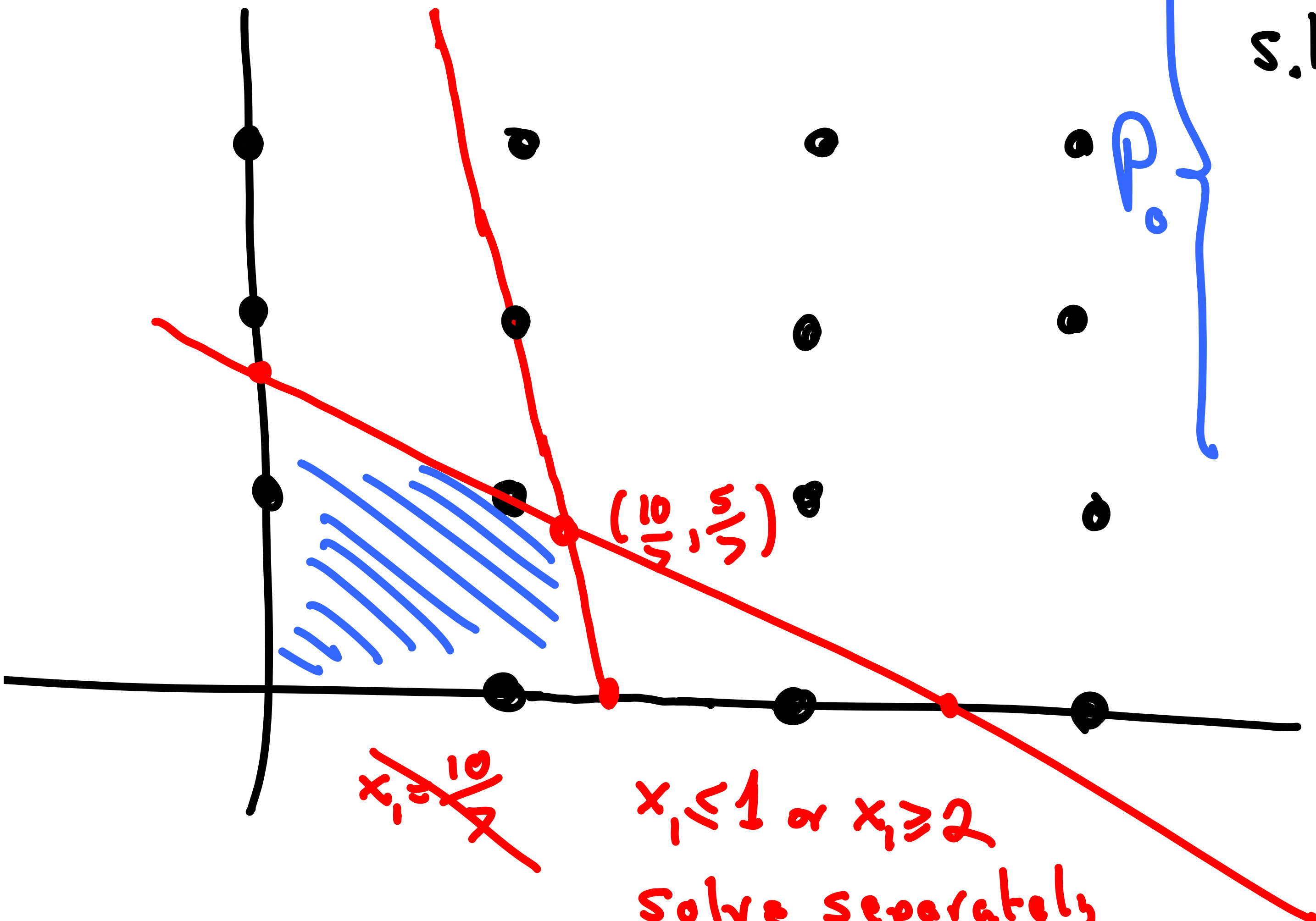
$$x_0 + \frac{19}{14}x_3 + \frac{1}{14}x_4 = \frac{50}{7} \Rightarrow \frac{5}{14}x_3 + \frac{1}{14}x_4 \geq \frac{1}{7}$$

| B.V. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $\xi_1$ | R |
|---|---|---|---|---|---|---|
| $x_0$ | | | 1 | | 1 | 7 |
| $x_2$ | | 1 | 1 | | -2 | 1 |
| $x_1$ | 1 | | -1 | | 3 | 1 |
| $\xi_1$ | | | 5 | 1 | -14 | 2 |

<u>Integer Optimum.</u>

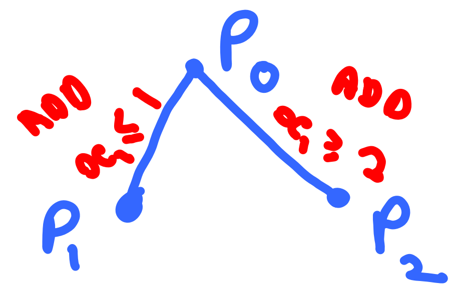
# Branch and Bound

$$P_0 \left\{ \begin{array}{ll} \text{Maximize} & 3x_1 + 4x_2 \\ \text{s.t.} & 2x_1 + 3x_2 \le 5 \\ & 4x_1 - x_2 \le 5 \\ & x_1, x_2 \ge 0 \\ & \text{\& integer.} \end{array} \right.$$

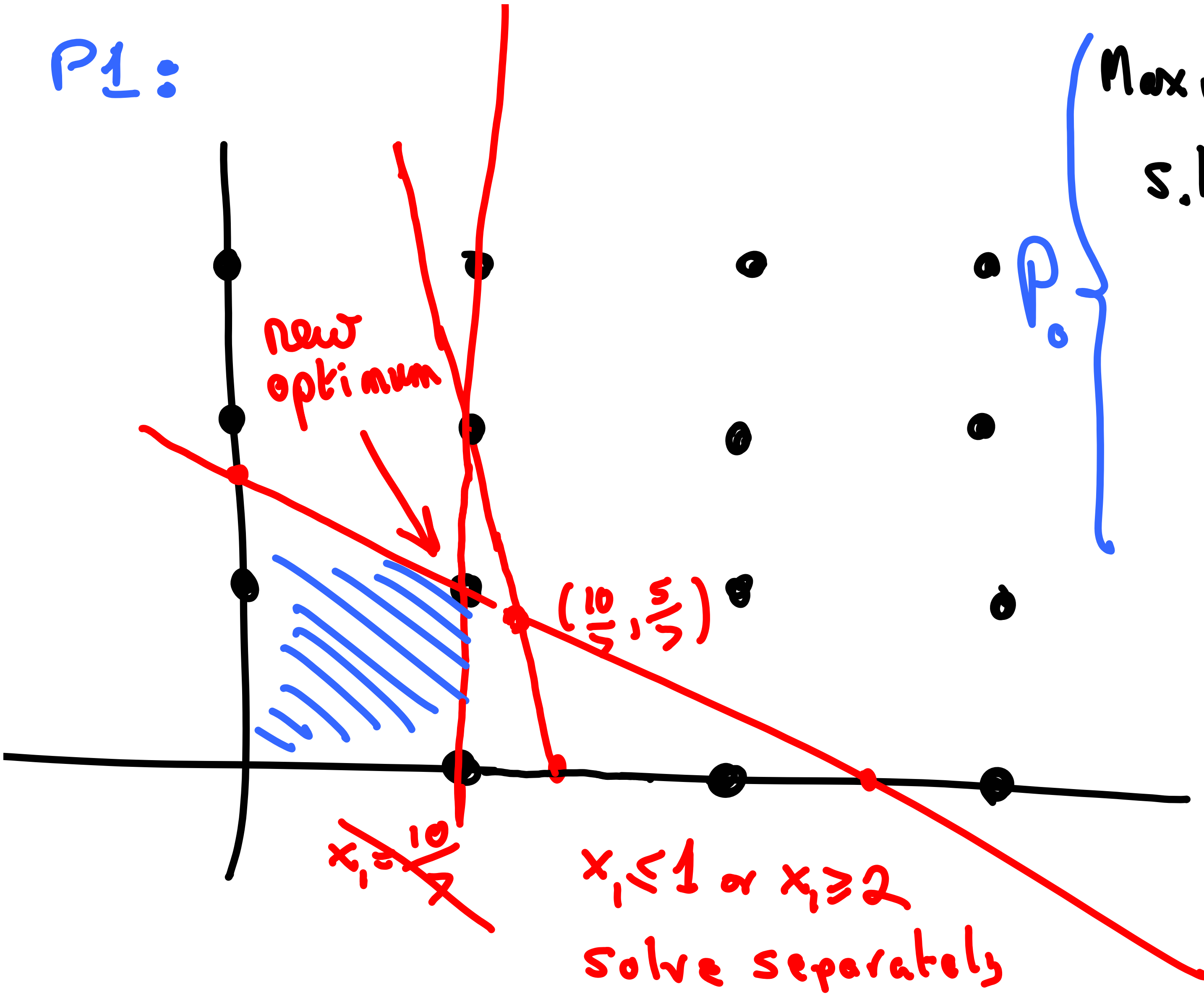



$x_1 \le 1$ or $x_1 \ge 2$

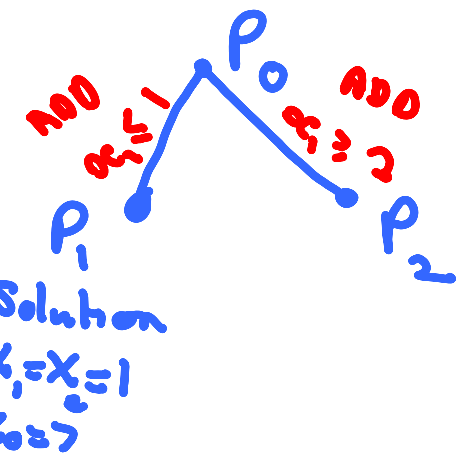
Solve separately



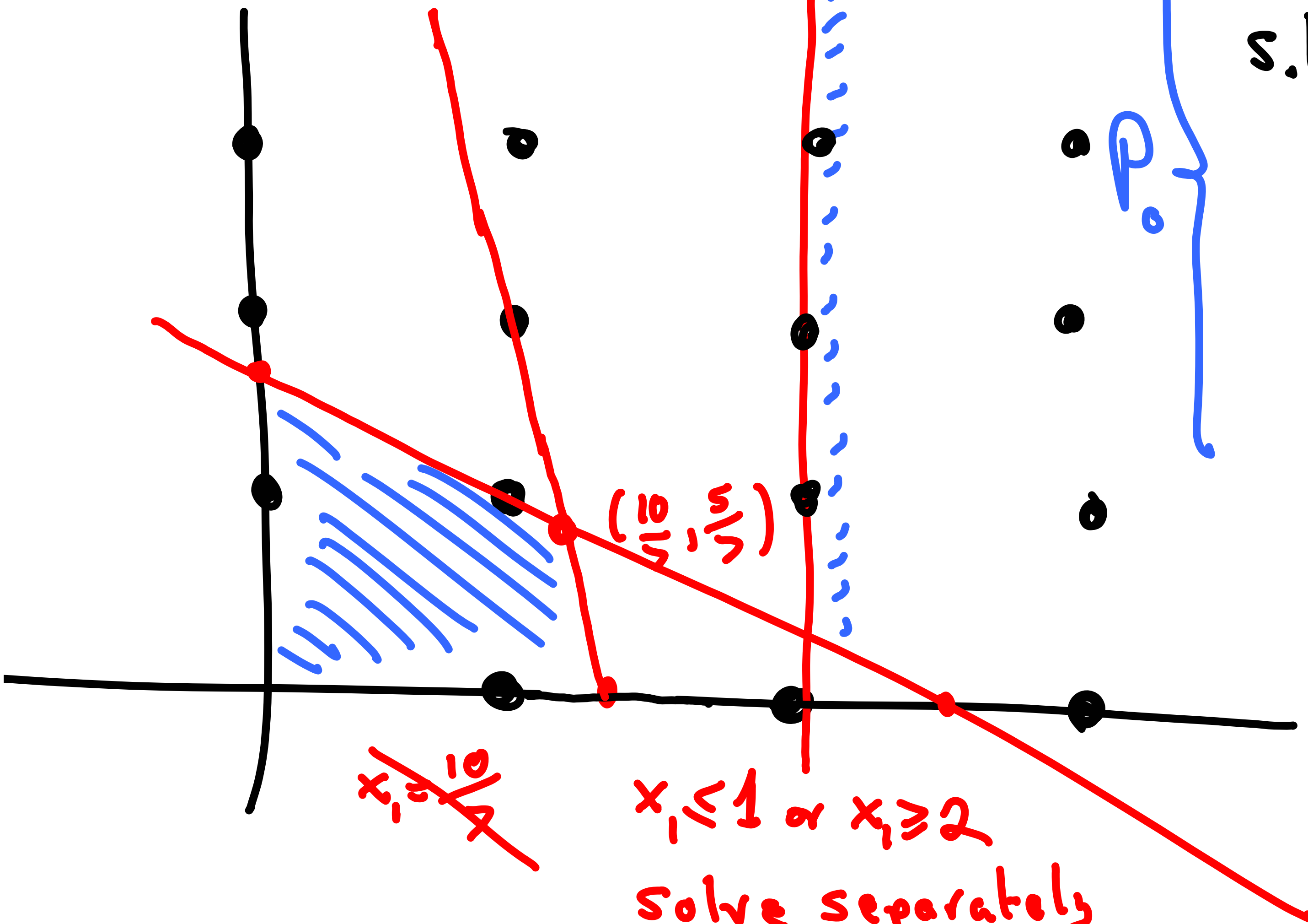
P1:



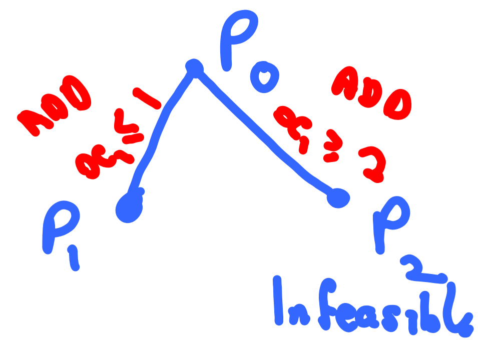

$$P_0 \left\{ \begin{array}{ll} \text{Maximise} & 3x_1 + 4x_2 \\ \text{s.t.} & 2x_1 + 3x_2 \leq 5 \\ & 4x_1 - x_2 \leq 5 \\ & x_1, x_2 \geq 0 \\ & \text{\& integer.} \end{array} \right.$$



P2:



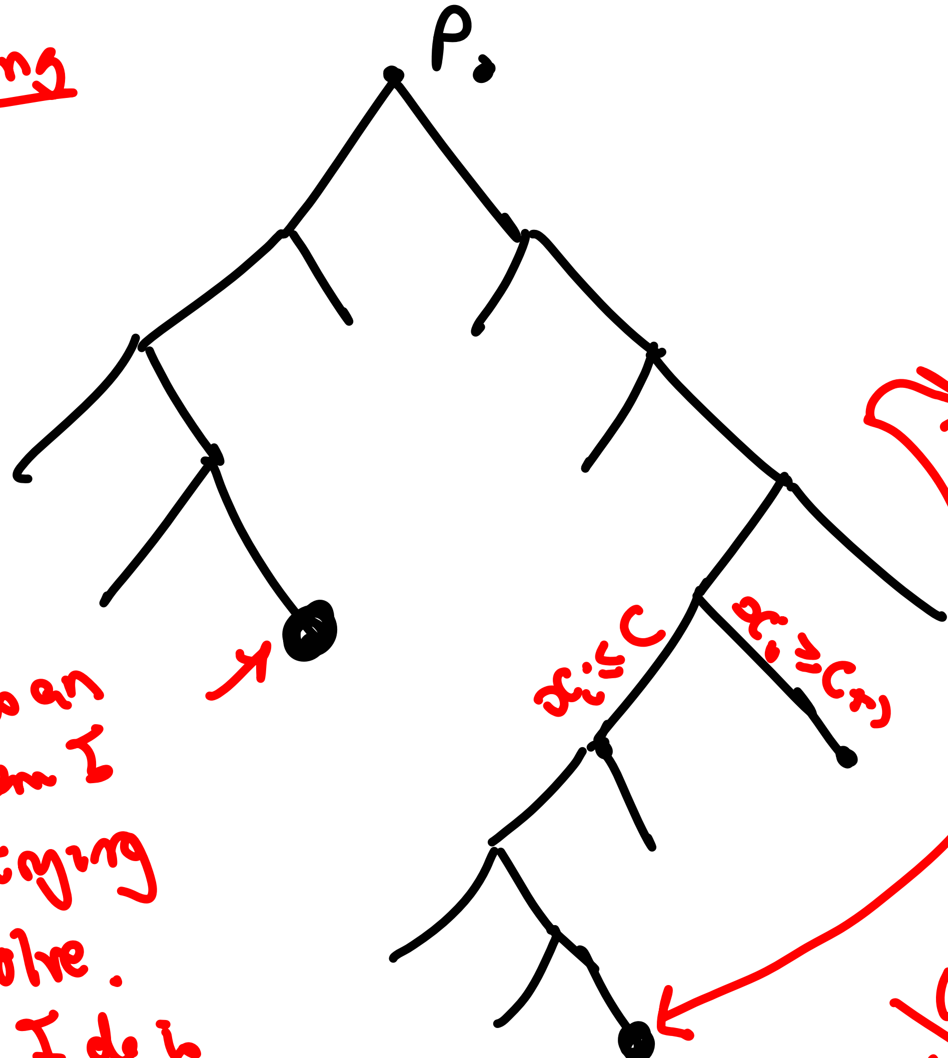

$$P_0 \left\{ \begin{aligned} \text{Maximize} \quad & 3x_1 + 4x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \le 5 \\ & 4x_1 - x_2 \le 5 \\ & x_1, x_2 \ge 0 \\ & \text{\& integer.} \end{aligned} \right.$$



P1 : Optimum value 7 $\longleftarrow$ Optimum Overall.

$P_2$ : infeasible

# Maximising




This is an LP problem I am trying to solve.

First thing I do is solve LP relaxation.

## Possibilities

(I) LP infeasible
Stop branching from here

(II) LP solution is integer.
Compare with best we have found so far.
Best value so far is $Z^*$

(III) LP value $Z_{LP} \le Z^*$
Stop branching

(IV) $Z_{LP} > Z^*$ : Split problem

# Implicit Enumeration

Branch and bound without solving LP.

Weaker bounds but less computation per node.

Maximize $5x_1 + 4x_2 + 3x_3 - x_4 + 3x_5$

$x_1, \dots x_5 = 0$ or $1$.

$$x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 \leq 4$$

$$-x_1 + x_2 - x_3 + x_4 - x_5 \geq 2$$

$$2x_1 - x_2 + x_3 - x_4 + x_5 \leq 3$$

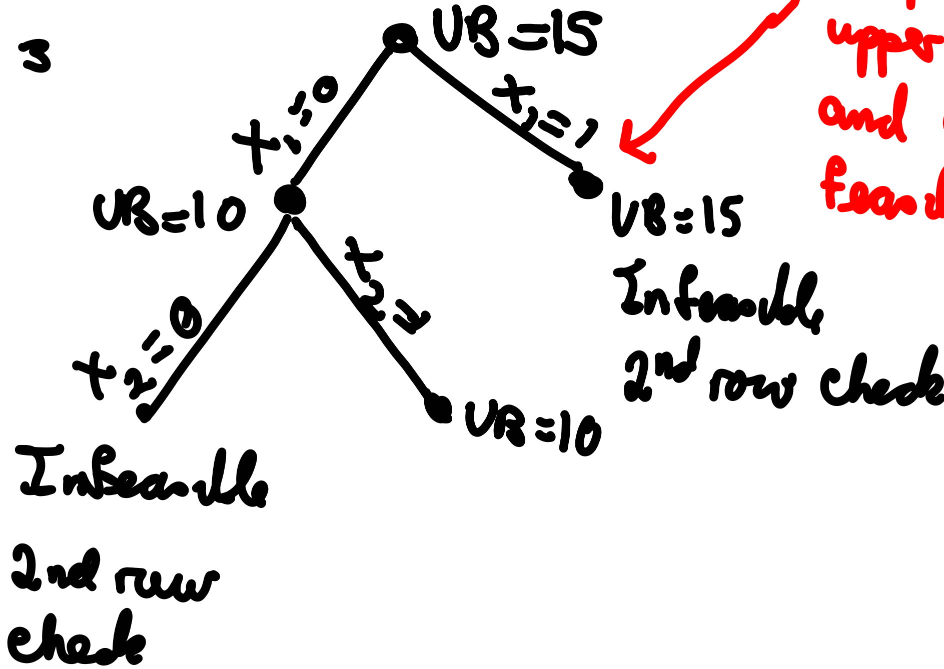
Maximize $6x_1 + 4x_2 + 3x_3 - x_4 + 3x_5$

$x_1, \dots, x_5 = 0$ or $1$.

$$x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 \leq 4$$
$$-x_1 + x_2 - x_3 + x_4 - x_5 \geq 2$$
$$2x_1 - x_2 + x_3 - x_4 + x_5 \leq 3$$

Upper Bound UB

Feasibility check:
Is it obviously infeasible i.e.
is there a row that cannot be satisfied?




compute an upper bound and check feasibility.