Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 2: Due Monday October 25.

1. An assembly line consists of a sequence of locations called work stations. The manufacture of a certain object requires m separate jobs to be undertaken with job i requiring t_i minutes. The jobs are to be allocated to work stations so that each station completes a set of jobs and then passes the object onto the next station on the line and waits to receive the next object from the previous station on the line. The combined time of all jobs assigned to any station must not exceed T the cycle time. Also there are a number of precedence relations between jobs indicated by the digraph $D = (V, A)$ where $(i, j) \in A$ if job i must precede job j. The problem is to open as few work stations as possible consistent with the cycle time.

   Solution

$$min \sum_{i=1}^{m} y_i$$

$$\begin{array}{rcl}
s.t. & & \\
x_{ij} & \leq & y_i \quad \forall i, j \in \{1, \ldots, m\} \\
\sum_{j=1}^{m} x_{ij} t_j & \leq T & \forall i \in \{1, 2, \ldots, m\} \\
\sum_{i=1}^{m} x_{ij} & = & 1 \quad \forall j \in \{1, 2, \ldots, m\} \\
\sum_{i=1}^{m} i x_{ij_1} & \leq & \sum_{i=1}^{m} i x_{ij_2} \quad \forall (j_1, j_2) \in A \\
x_{ij} & \in & \{0, 1\} \forall i, j \in \{1, \ldots, m\} \\
y_i & \in & \{0, 1\} \forall i \in \{1, \ldots, m\}
\end{array}$$

   x_{ij} is 1 when job j is done at station i and 0 otherwise.

   y_i is 1 if at least one job is done at station i.

The first constraint ensures that if any job is done at station i, the variable y_i is 1.

The second constraint ensures that each station satisfies the cycle time T.

The third constraint ensures that each job is scheduled on some machine.

The last constraint ensures the j_1 is done before job j_2 if there is a precedence constraint between them.

2. Solve the following problem by a cutting plane algorithm:

$$
\begin{array}{lcccccc}
\text{minimise} & 4x_1 & + & 5x_2 & +3x_3 & & \\
\text{subject to} & & & & & & \\
 & 2x_1 & + & x_2 & -x_3 & \geq & 2 \\
 & x_1 & + & 4x_2 & +x_3 & \geq & 13
\end{array}
$$

$$x_1, x_2, x_3 \geq 0 \text{ and integer.}$$

Solution

Initial tableau

x_1	x_2	x_3	x_4	x_5	R.H.S	
-4	-5	-3	0	0	0	z
-2	-1	1	1	0	-2	x_4
-1	-4	-1	0	1	-13	x_5 $\leftarrow$
	$\uparrow$					

x_1	x_2	x_3	x_4	x_5	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	$\frac{13}{4}$	x_2

Primal feasible, but the solution is not integral.

We add a cut which eliminates the current optimal solution.

$$\frac{1}{4}x_1 + \frac{1}{4}x_3 + \frac{3}{4}x_5 - y_1 = \frac{1}{4}$$

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-11}{4}$	0	$\frac{-7}{4}$	0	$\frac{-5}{4}$	0	$\frac{65}{4}$	z
$\frac{-7}{4}$	0	$\frac{5}{4}$	1	$\frac{-1}{4}$	0	$\frac{5}{4}$	x_4
$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{-1}{4}$	0	$\frac{13}{4}$	x_2
$\frac{-11}{4}$	0	$\frac{-1}{4}$	0	$\frac{-3}{4}$	$+1$	$\frac{-1}{4}$	$y_1 \leftarrow$
				$\uparrow$			

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_1	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	$\frac{-5}{3}$	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	$\frac{-1}{3}$	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{-1}{3}$	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	$\frac{-4}{3}$	$\frac{1}{3}$	x_5

The solution is primal feasible and so optimal but still not integer.

We add a cut which eliminates the current optimal solution.

$$\frac{-1}{3}x_1 - \frac{1}{3}x_3 + y_2 = \frac{1}{3}$$

We obtain tableau

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
$\frac{-7}{3}$	0	$\frac{-4}{3}$	0	0	0	$\frac{50}{3}$	z
$\frac{-5}{3}$	0	$\frac{4}{3}$	1	0	0	$\frac{4}{3}$	x_4
$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	0	$\frac{10}{3}$	x_2
$\frac{1}{3}$	0	$\frac{1}{3}$	0	1	0	$\frac{1}{3}$	x_5
$\frac{-1}{3}$	0	$\frac{-1}{3}$	0	0	1	$\frac{1}{3}$	$y_2 \leftarrow$
		$\uparrow$					

We do a dual simplex pivot to obtain

x_1	x_2	x_3	x_4	x_5	y_2	R.H.S	
-1	0	0	0	0	-4	18	z
-3	0	0	1	0	4	0	x_4
0	1	0	0	0	1	3	x_2
0	0	0	0	1	1	0	x_5
1	0	1	0	0	-3	1	x_3

Which is optimal integral.

3. Solve the following problem by a branch and bound algorithm:

$$\begin{array}{llllll}
\text{Maximise} & 4x_1 & -2x_2 & +7x_3 & -x_4 & \\
\text{subject to} & & & & & \\
& x_1 & & +5x_3 & & \leq 10 \\
& x_1 & +x_2 & -x_3 & & \leq 1 \\
& 6x_1 & -5x_2 & & & \leq 0 \\
& -x_1 & & +2x_3 & -2x_4 & \leq 3
\end{array}$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

$$x_1, x_2, x_3 \text{ integer.}$$

Solution

1. LP relaxation:

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 0\right) \qquad Value = 14\frac{1}{4}.$$

Sub-problem 1: add constraint $x_1 \leq 1$.

$$(x_1, x_2, x_3, x_4) = \left(1, \frac{6}{5}, \frac{9}{5}, 0\right) \qquad Value = 14\frac{1}{5}.$$

Sub-problem 2: add constraint $x_1 \geq 2$.
No solutions.

Subproblem 1.1: add constraint $x_2 \leq 1$.

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 1, \frac{11}{6}, 0\right) \qquad Value = 14\frac{1}{6}.$$

Subproblem 1.2: add constraint $x_2 \geq 2$.

$$(x_1, x_2, x_3, x_4) = \left(\frac{5}{6}, 2, \frac{11}{6}, 0\right) \qquad Value = 12\frac{1}{6}.$$

Sub-problem 1.1.1: add constraint $x_1 \leq 0$.

$$(x_1, x_2, x_3, x_4) = \left(0, 0, 2, \frac{1}{2}\right) \qquad Value = 13\frac{1}{2}.$$

This solution is feasible.

Subproblem 1.1.2: add constraint $x_1 \geq 1$. No solutions.

Sub-problem 1.2 is *fathomed* i.e. there is no solution to this problem which is better than our current *incumbent*.

Optimal solution: $(x_1, x_2, x_3, x_4) = \left(0, 0, 2, \frac{1}{2}\right) \qquad Value = 13\frac{1}{2}$.