9\27\10

Game Theory - 2 person, zero-sum

Game 1.

2 players A, B

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & 3 & -2 & 4 \\ 5 & 2 & 1 & -3 \\ 4 & 3 & -1 & 4 \end{bmatrix} \end{array}$$

Sum of winnings equals zero

A chooses $i = 1, 2, 3$

B chooses $j = 1, 2, 3, 4$

B pays A, a_{ij}

Choices are made independently e.g. A writes i on a piece of paper, B writes j. C looks at the 2 piece of paper.

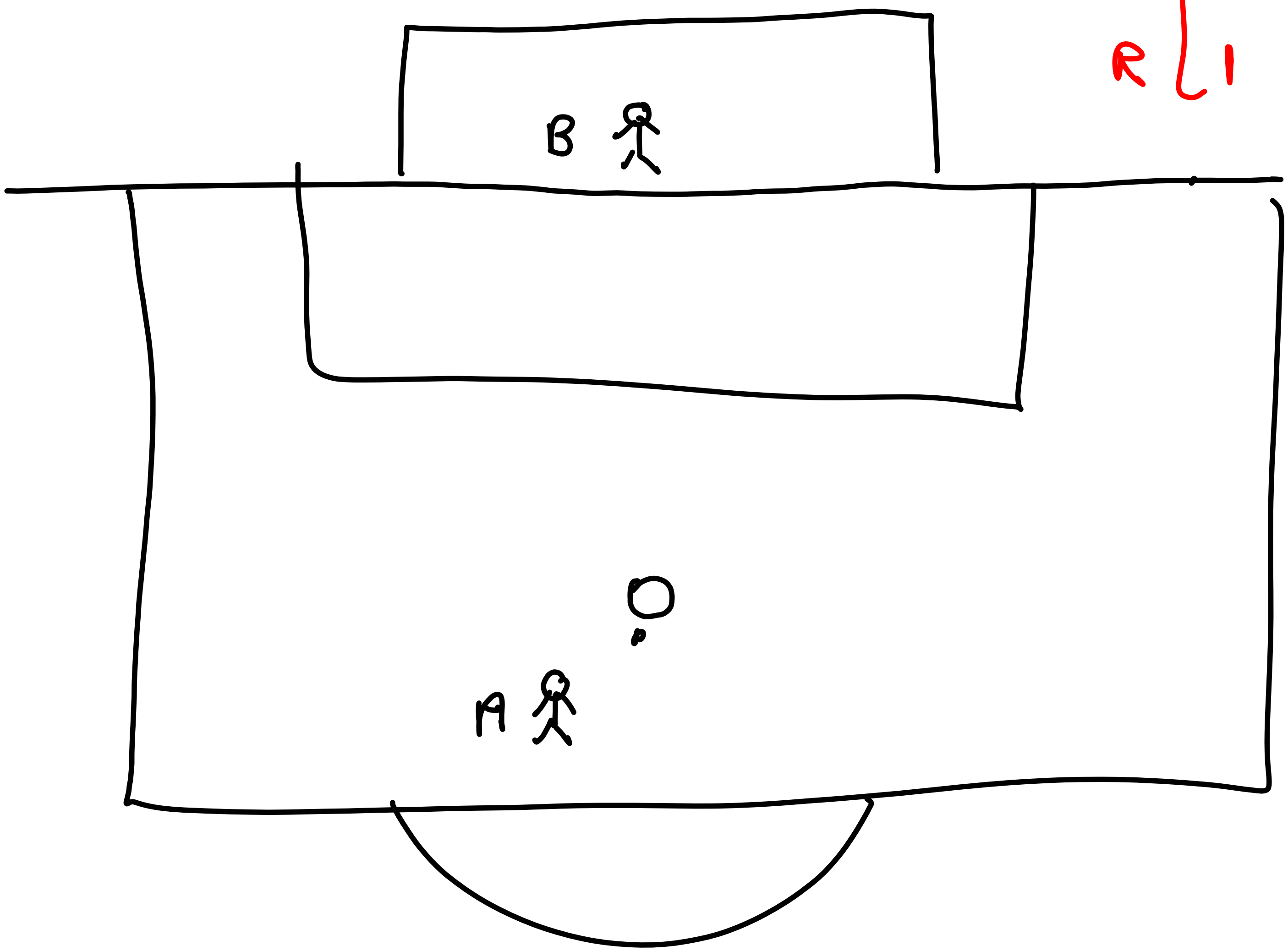
Examples

Rock, Paper, Scissors

$$\begin{array}{c c} & \begin{array}{ccc} R & P & S \end{array} \\ \begin{array}{c} R \\ P \\ S \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

Penalty Kicks : Soccer

$$
\begin{array}{c|ccc}
A \backslash B & S & L & R \\
\hline
S & & & \\
L & & & \\
R & & &
\end{array}
\quad
\begin{array}{c} \\ S \\ L \\ R \end{array}
\begin{array}{c} S \quad\; L \quad\; R \\ \begin{bmatrix} -1 & 2 & 2 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{bmatrix} \end{array}
$$



play: one execution of the game.

match: unending sequence of plays

Strategy: a rule for next choice

S_A, S_B are the sets of strategies for A, B

Initially $S_A = \{ (i) : i = 1, 2, \dots, m \}$

$(i) = i, i, i, i, \dots \dots$

pure strategy

$S_B = \{ (j) \}$

For $u \in S_A$, $v \in S_B$: $PAY(u, v)$ = average payoff

$$\text{PAY}\big((i),(j)\big) = a_{ij}$$

We assume both players are conservative

P_A = minimum that A can guarantee to win.

$$\text{ROWMIN}(u) = \min_{v} \text{PAY}(u,v)$$

$$P_A = \max_{u} \text{ROWMIN}(u)$$

$$\text{COLMAX}(v) = \max_{u} \text{PAY}(u,v)$$

$P_B = \min\limits_{v} \text{COLMAX}(v)$ = maximum that B will have to lose

$$\mathrm{ROWMIN}(u) \leq \mathrm{PAY}(u,v) \leq \mathrm{COLMAX}(v)$$

$$P_A \leq P_B$$

Suggestion: A choose u to maximise ROWMIN

B choose v to minimise COLMAX

$P_A = -1$

$P_B = 4$

$$\begin{array}{cc} & \begin{array}{ccc} * & & \end{array} \\ \begin{array}{c} \\ \\ * \end{array} & \begin{bmatrix} 4 & 3 & -2 \\ 2 & -5 & 5 \\ -1 & 6 & 3 \end{bmatrix} \end{array}$$

Stable Solution

(u_0, v_0) is stable

$$PAY(u, v_0) \leq PAY(u_0, v_0) \leq PAY(u_0, v)$$

Neither player has an incentive to change if he/she thinks other player won't.

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & 6 \\ -3 & -2 & 5 \end{bmatrix}$$

$P_A = 1$

$P_B = 1$

CLAIM

A stable solution exists iff $P_A = P_B$.

Proof

(1) Suppose (u_0, v_0) is stable

$$\mathrm{PAY}(u, v_0) \le \mathrm{PAY}(u_0, v_0) \le \mathrm{PAY}(u_0, v)$$

$$\mathrm{COLMAX}(v_0) = P_B \qquad \mathrm{ROWMIN}(u_0) = P_A$$

(11) $P_A = P_B$

$\mathrm{ROWMIN}(u_0)$ $\mathrm{COLMAX}(v_0)$

(u_0, v_0) is stable

$\mathrm{PAY}(u_0, v_0)$

$P_B \geq \mathrm{PAY}(u_0, v_0) \geq P_A = P_B$