10\20\10

Shortest path when arc lengths can be negative.

Warning: need some conditions.

Suppose I want to know whether or not an n-vertex digraph has a Hamilton path.
[A path of length $n-1$ through every vertex]

If I make $\ell(i,j) = -1$ for $(i,j) \in E$ & 0 otherwise

then min length path in graph is $-(n-1)$ iff $\exists$ Hamilton path

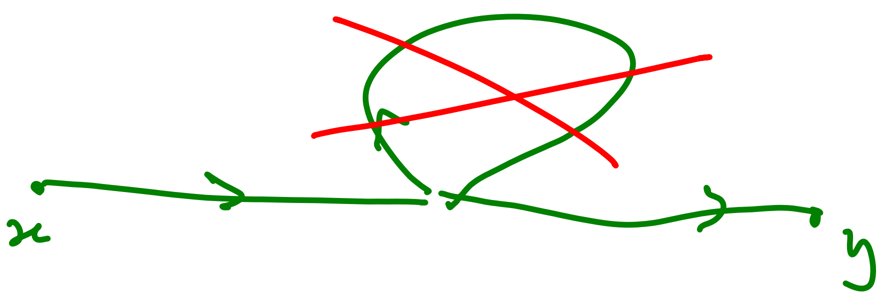
Extra condition:

$$\ell(C) \geq 0 \quad \text{for all cycles } C.$$

$$\left(\ell(C) = \sum_{(i,j)\in C} \ell(i,j)\right.$$

So shortest path $x \to y$

$\equiv$ shortest walk $x \to y$



Finding shortest walk $x \to y$ is doable

Optimality Condition

Suppose we want to find a shortest walk from s to every vertex.

Suppose that for all $v \in V$ we have a walk $W(v)$ of length $d(v)$ from s to v.

<u>Claim</u> These walks constitute a set of shortest walks from s iff

$$d(w) \leq d(v) + \ell(v,w)$$

$$\forall v, w$$

(I) If $d(w) > d(v) + \ell(v, w)$

then $\ell(W(w)) > \ell(W(v), w)$.

(II) Suppose condition holds.

Suppose $W = (w_0 = s, w_1, w_2, \ldots, w_k = v)$ is a walk from s to v.

We need to prove that $\ell(W) \geq d(v)$

$$
\begin{aligned}
\ell(W) = \;& \ell(w_0, w_1) && \geq d(w_1) - d(s) \\
&+ \ell(w_1, w_2) && \geq d(w_2) - d(w_1) \\
&+ \\
&\;\vdots && \;\vdots \\
&+ \ell(w_i, w_{i+1}) && \geq d(w_{i+1}) - d(w_i) \\
&\;\vdots && \;\vdots \\
&+ \ell(w_{k-1}, v) && \geq d(v) - d(w_{k-1}) \\
\hline
& && d(v) - d(s)
\end{aligned}
$$

$$\ell(W) \geq d(v) - \underset{0}{\underset{\shortparallel}{d(s)}}$$

<u>Algorithm</u>

Start with some walks $W(v)$, $v \in V$

$d(v) = \ell(W(v)) \quad \forall v \in V$

<u>repeat</u>

If $\exists\, (v, w)$ such that

$$d(w) > d(v) + \ell(v, w)$$

then replace $W(w)$ by $(W(v), w)$

$d(w) > d(v) + \ell(v, w)$

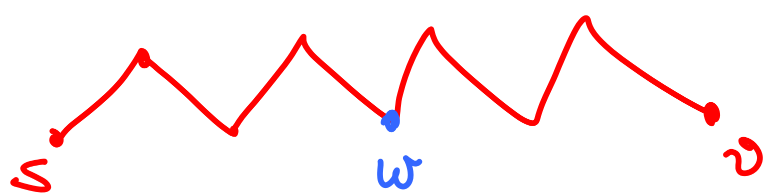
<u>until</u> optimality condition holds.

If the algorithm terminates then it has found shortest walks.

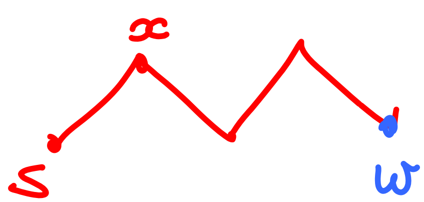
Refine Algorithm

1) We can start with $d(s) = 0$. [$d(s)$ is correct immediately

2) Suppose



shortest s to v



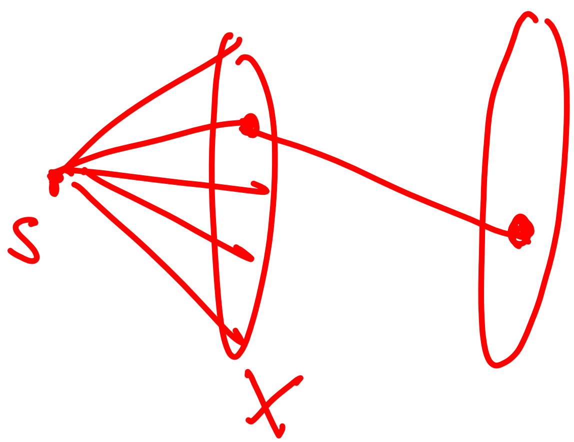
shortest s to w

So edge (s,x) is a shortest walk to x

3. Let $X = \{x : (s,x) \text{ is shortest walk to } x\}$.

If we go through each edge (s,v), $v \in V$ in turn then $d(x) = l(s,x)$ will be correct for $x \in X$.

$0 + l(s,x) \quad v \quad d(x)$



Algorithm

$E = \{e_1, e_2, \ldots, e_m\}$ = edges of Digraph

$e_i = (x_i, y_i)$

repeat:

ROUND

for $i = 1, 2, \ldots, m$

if $d(y_i) > d(x_i) + l(e_i)$

then

$d(y_i) \leftarrow d(x_i) + l(e_i)$

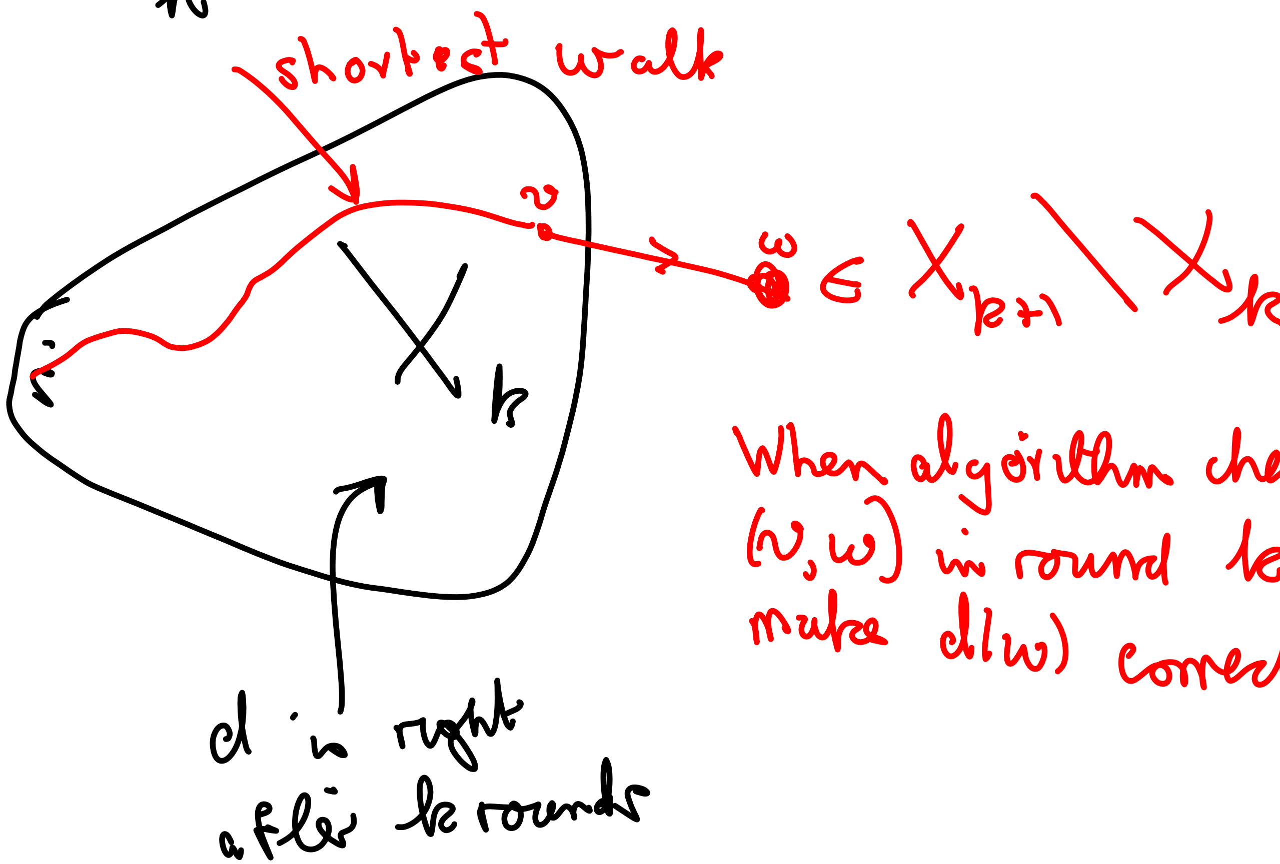
$W(y_i) \leftarrow (W(x_i), y_i)$

until optimality condition

rounds $\leq n-1$

After k rounds $d(x)$ is correct $\forall x$ such $\exists$ shortest walk $s \to x$ with $\leq k$ edges

$X_k := \{ v : \exists \text{ shortest walk}, \leq k \text{ edges} \}$



When algorithm checks (v, w) in round $k+1$ it make $d(w)$ correct.