10\13\10

Shortest Path Problem

Digraph $D = (V, A)$

$l : A \longrightarrow \mathbb{R}$ [edge a has length $l(a)$]

If P is a path

$$l(P) = \sum_{a \in P} l(a)$$

Problem:

Single Source Problem:

Given $s \in V$, find a shortest path from s to all $v \in V$.

All Pairs Problem:

Find a shortest path between every pair of vertices.

We can assume $l(u) \geq 0, \forall u \in A$

or

we can allow some arcs to have $l(u) < 0$.

(1) Non-negative arc lengths.

Dijkstra's Algorithm:

Initially $d(s) = 0$ & $d(v) = \infty$, $s \neq v$

$x = s$; $S = \{x\}$,

repeat

Process x;

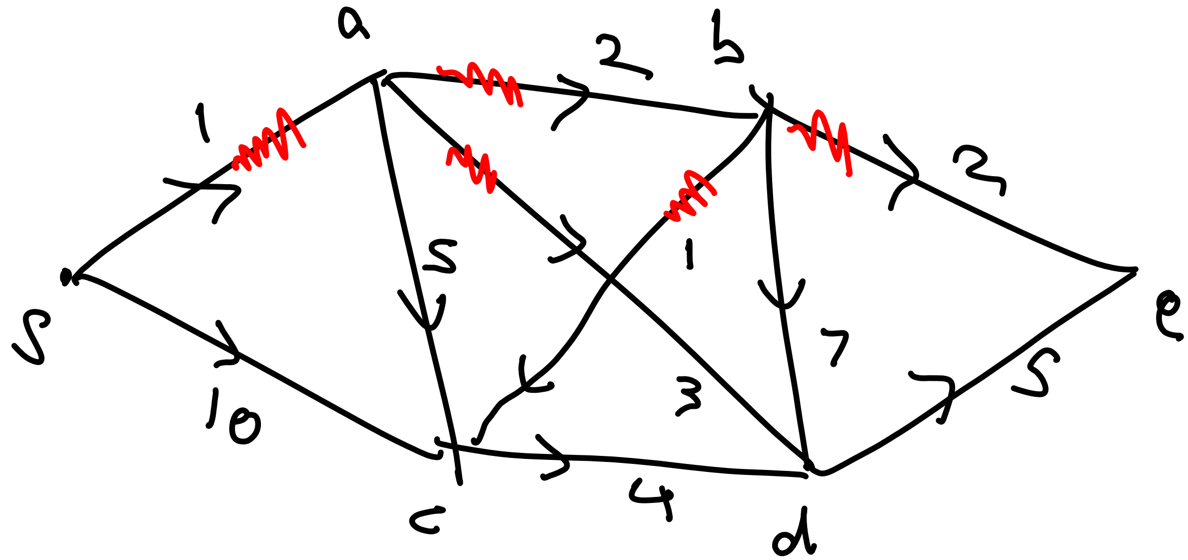
$\forall v: d(v) := \min(d(v), d(x) + \ell(x, v))$

$S := S \cup \{y\}, d(y) = \min_{u \notin S} d(u)$

$x := y$

until $S = V$

S



	s	a	b	c	d	e
d	$\boxed{0}$	∞	∞	∞	∞	∞
d	$\boxed{0}$	$\boxed{1}$	∞	10	∞	∞
d	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	6	4	∞
d	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{4}$	4	5
d	$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{4}$	4	5

Time Dependent Arc Lengths

Replace $\ell : A \longrightarrow \mathbb{R}^+$

by $\ell : A \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

$u \in A$ then $\ell(u,t)$ = time taken to
cross u starting
at time t.

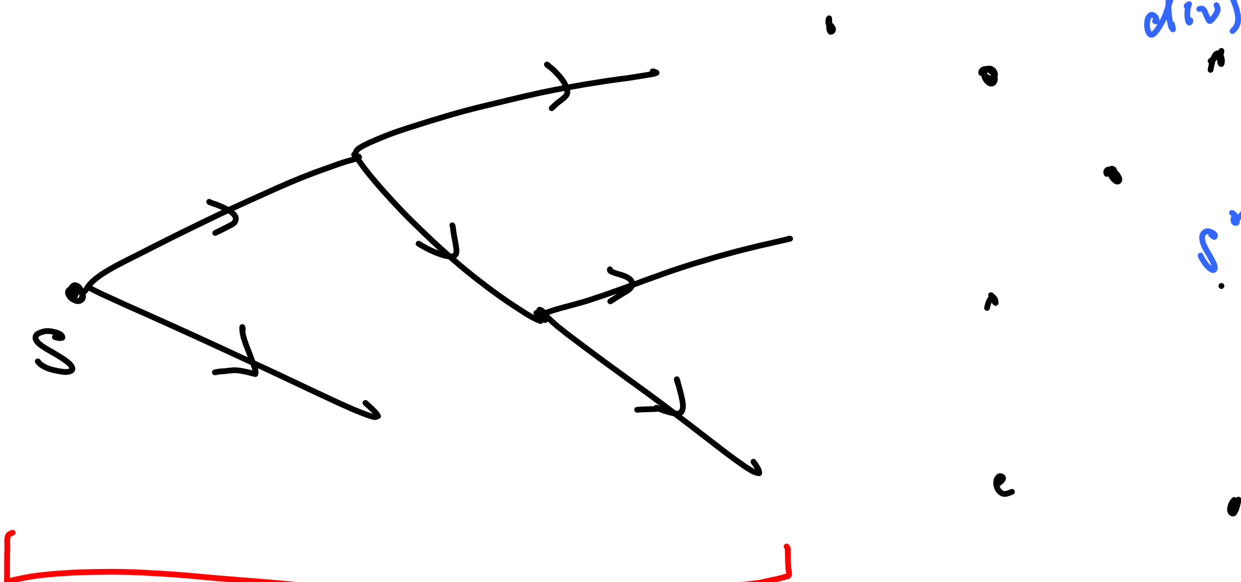
$P = \{u_0, u_1, u_2, \ldots u_k$

$$\ell(P) = \underbrace{\ell((u_0,u_1),0)}_{\xi} + \ell((u_1,u_2),\xi) + \ldots$$

$$P = u_0, u_1, u_2, \dots u_k$$

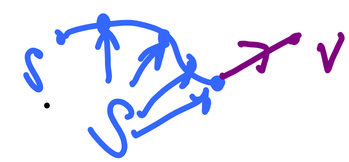
$$\ell(P) = \underbrace{\ell\big((u_0, u_1), 0\big)}_{\xi_1} + \underbrace{\ell\big((u_1, u_2), \xi_1\big)}_{\xi_2} + \ell\big((u_2, u_3), \xi_1 + \xi_2\big) + \dots$$

Dijkstra's Algorithm can be adapted to this situation:



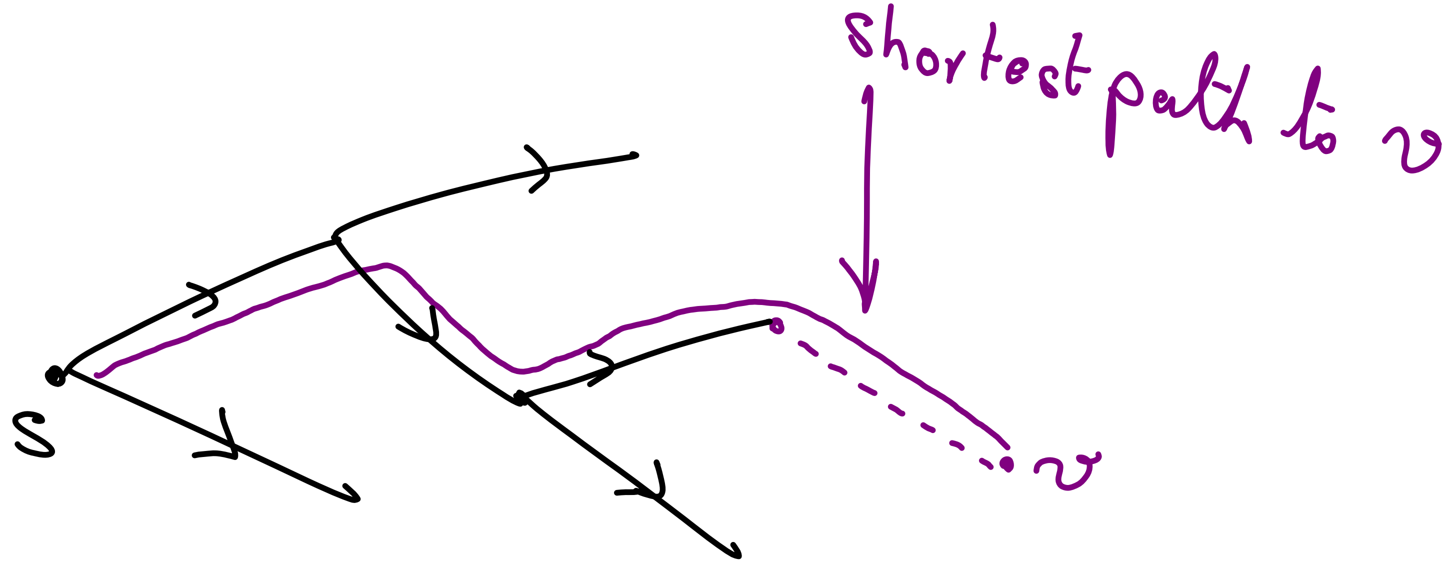
S

$d(v)$ = min length path of the form



claim: $v \in S \Rightarrow d(v)$ = length of shortest path from s to v

True initially



Any other path to v