9\21\09

LP approach:

Maximise $y_1 + y_2 + \cdots + y_n$

s.t

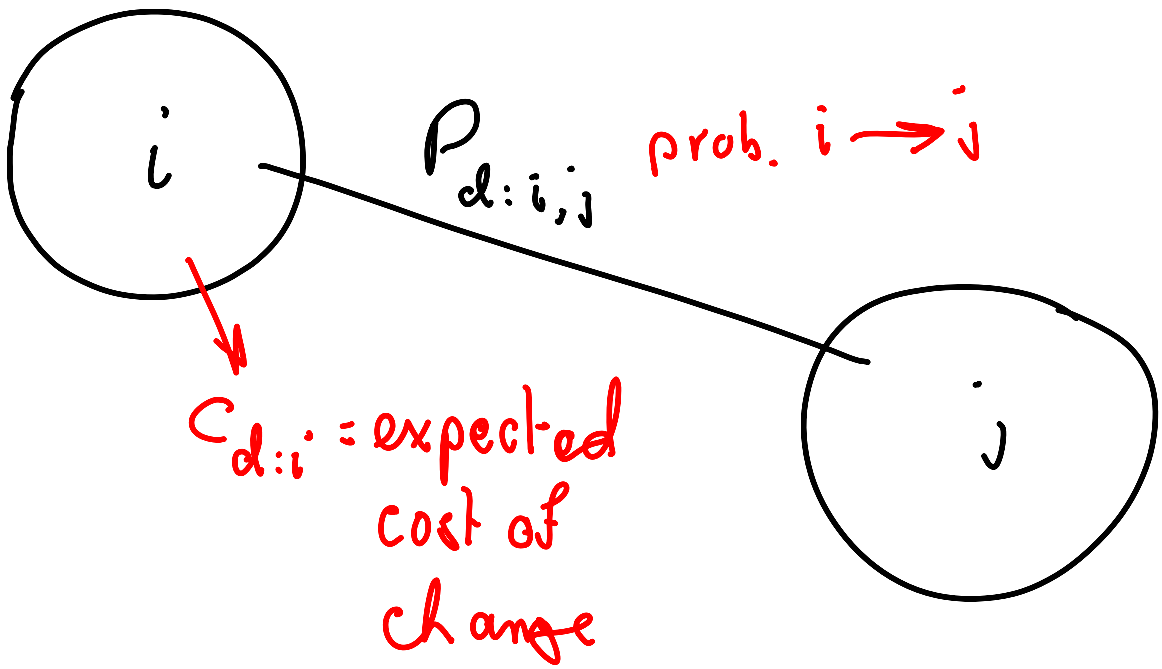
$$y_i \leq c_{ij} + \alpha y_j \quad , \quad \forall i, j$$

At a maximum $y_i = \min_j \; c_{ij} + \alpha y_j \quad \forall i$

optimality conditions hold.

Add probability:

$d \in D_i$



Policy: $\pi(i) \in D(i), \ \forall i$

y_i = expected discounted cost of continuing indefinitely

$$= c_{\pi(i):i} + \alpha \sum_{j \in V} P_{\pi(i):i,j}\, y_j$$

$$\underline{y} = \underline{c}_\pi + \alpha P_\pi \underline{y} \qquad \underline{y} = n \times 1$$

$$(I - \alpha P_\pi)\,\underline{y} = \underline{c}_\pi$$

$$\underline{y} = (I - \alpha P_\pi)^{-1}\, \underline{c}_\pi$$

$$(I - \alpha P_\pi)(I + \alpha P_\pi + \alpha^2 P_\pi^2 + \cdots + \alpha^k P_\pi^k)$$

$$= I - \underbrace{\alpha^{k+1} P_\pi^{k+1}}_{\downarrow \; 0}$$

← (i, j) th entry = Prob. of going i to j in k+1 steps

$$(I - \alpha P_\pi)^{-1} = I + \alpha P_\pi + \alpha^2 P_\pi^2 + \cdots + \alpha^k P_\pi^k + \cdots$$

$$\geq 0$$

Optimality Criterion

$$y_i = c_{\pi(i),i} + \alpha \sum_{j \in V} P_{\pi(i);i,j}\, y_j$$

$$= \min_{d \in D_i} \left\{ c_{d,i} + \alpha \sum_{j \in V} P_{d:i,j}\, y_j \right\} \quad (*)$$

Proof

Suppose (*) holds and $\hat{\pi}$ is another policy

$$\hat{y}_i = c_{\hat{\pi}(i),i} + \alpha \sum_{j \in V} P_{\hat{\pi}(i):i,j}\, \hat{y}_j$$

$$y_i \leq c_{\hat{\pi}(i),i} + \alpha \sum_{j \in V} P_{\hat{\pi}:i,j}\, y_j$$

$$\underline{\hat{y}} - \underline{y} \geq \alpha P_{\hat{\pi}} (\underline{\hat{y}} - \underline{y})$$

$$\underline{\hat{y}} - \underline{y} \geq \alpha P_{\hat{\pi}} (\underline{\hat{y}} - \underline{y})$$

$$(I - \alpha P_{\hat{\pi}})(\underline{\hat{y}} - \underline{y}) \geq 0$$

$(I - \alpha P_{\hat{\pi}})^{-1}$ only has ≥ 0 entries

$$(I - \alpha P_{\hat{\pi}})^{-1} (I - \alpha P_{\hat{\pi}})(\underline{\hat{y}} - \underline{y}) \geq 0$$

$$\underline{\hat{y}} - \underline{y} \geq 0$$

Suppose ⊛ does not hold.

$\hat{\pi}(i)$ minimizes $c_{\hat{\pi}(i),i} + \alpha \sum_{j \in V} P_{\hat{\pi}(i):i,j} \, y_j$

$\exists\, i$ where $y_i >$

$$y_i \geq c_{\hat{\pi}(i),i} + \alpha \sum_{j \in V} P_{\hat{\pi}(i):i,j} \, y_j$$

$$\hat{y}_i = c_{\hat{\pi}(i),i} + \alpha \sum_{j \in V} P_{\hat{\pi}(i):i,j} \, \hat{y}_j$$

$$\underline{y} - \underline{\hat{y}} \geq \alpha P_{\hat{\pi}} (y - \hat{y})$$

so $\underline{y} \geq \underline{\hat{y}}$

$\underline{y} \neq \underline{\hat{y}}$