11\04\09

Linear Programming Formulation of Assignment Problem:

$$x_{ij} = \begin{cases} 1 : \text{person } i \text{ does job } j \\ 0 : \text{otherwise} \end{cases}$$

Minimise $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j$$

$$\underbrace{x_{ij} = 0 \text{ or } 1}$$

LP relaxation: $0 \leqslant x_{ij}$

Claim: Any basic feasible solution to the LP-relaxation is an integer solution.

Constraint Matrix

$$\begin{array}{cl} \left[\begin{array}{c} 1 \\ \hline 1 \end{array}\right] & \begin{array}{l} n \text{ rows} \\ n \text{ rows} \end{array} \end{array}$$

Constraint Matrix

$$\begin{array}{c} \\ \text{times } -1 \end{array} \left[\begin{array}{c} 1 \\ \hdashline -1 \end{array}\right] \begin{array}{l} n \text{ rows} \\ n \text{ rows} \end{array}$$

Constraints:

$$A \underline{x} = \underline{b} \leftarrow \text{integer } \pm 1$$

every column $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Claim: every square sub-matrix B of A has determinant $0, \pm 1$

Totally Unimodular

Proof

Suppose that B is $k \times k$

Use induction on k.

$k=1$: trivial

Arbitrary k:

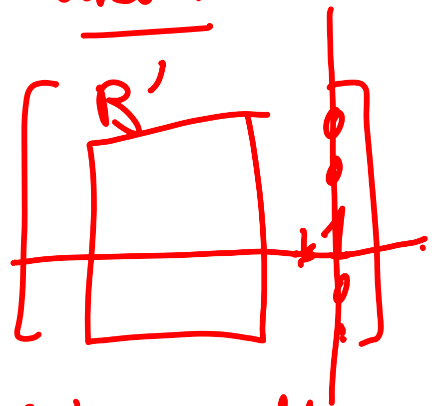
case 1

$$\begin{bmatrix} \text{every} \\ \text{column of } B \\ \text{has } 1 \text{ \& } -1 \end{bmatrix}$$

$\det B = 0$

sum of rows $= 0$

Case 2

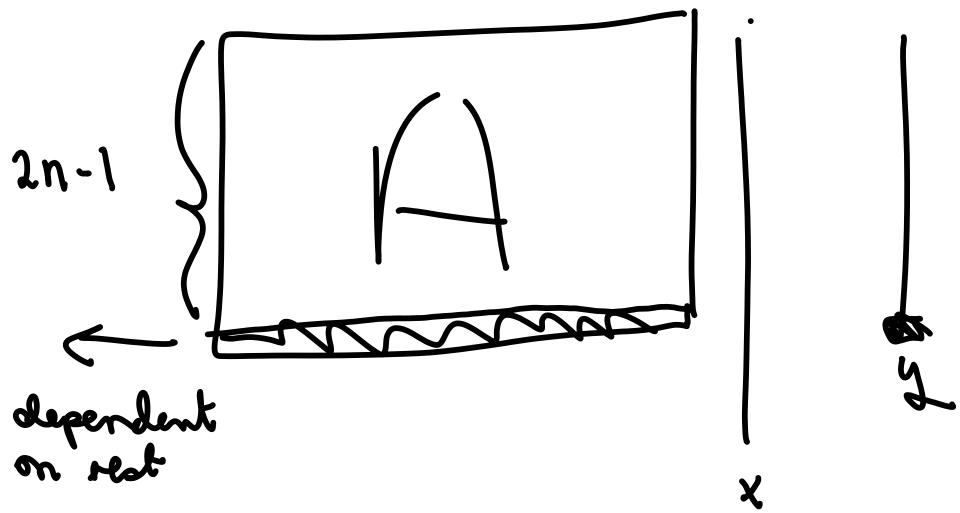


Column with unique non zero

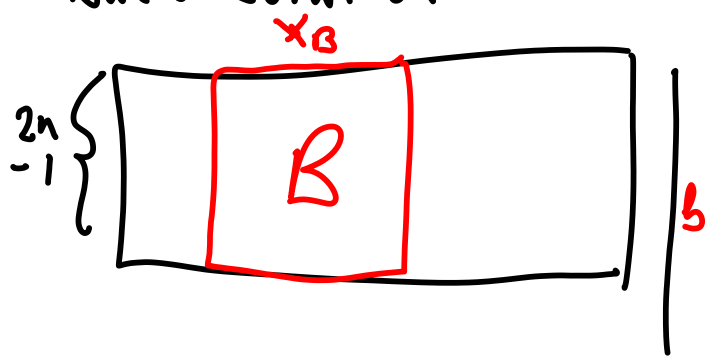
$$\det B = \pm \det B'$$
$$= \pm 1 \quad \text{induction}$$

Case 3

$$\begin{bmatrix} \text{all zero} \\ \text{matrix} \end{bmatrix} \quad \det = 0.$$



Basic solution



basic variables

$$x_B = \underbrace{B^{-1}}_{\text{integer matrix}} b$$

$$B^{-1} = \frac{\text{adj } B \leftarrow \text{integer}}{\det B \leftarrow \pm 1}$$

$$\begin{bmatrix} & \boxed{\begin{matrix} 1 \\ -1 \end{matrix}} & \end{bmatrix}$$

Graph G and oriented each edge

This collecting columns is lin. dep. iff edges contain a cycle.