10\30\09

Shortest Paths

Digraph $D = (N, A)$

$l : A \Rightarrow \mathbb{R}$ $\qquad$ $l(e)$ = length of edge e.

Path $P = (e_1, e_2, \dots, e_k)$ $\qquad$ $\vdots$ $\qquad$ $l(P) = \sum_{i=1}^{k} l(e_i)$

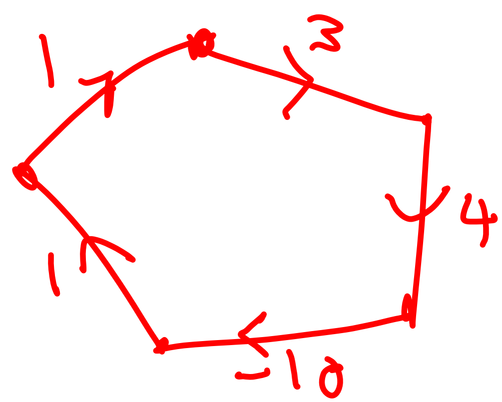
edges

Problem: given vertices s, t, find shortest path from s to t.

Arc lengths can be negative.

Well defined problem: # paths from s to t is $< n!$ $(n = |V|)$

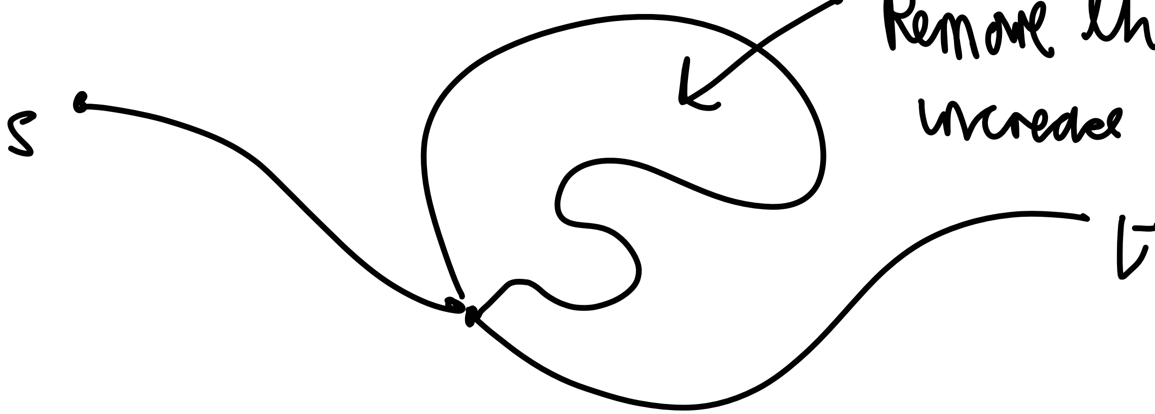
NP-hard in general, because of negative cycles



$\leftarrow$ length $= -1$

1) Algorithms are designed to find shortest walks

2) If there are no negative cycles then shortest walk $\equiv$ shortest path



Remove this without increase length.

3) If $\nexists$ negative cycles, then $\exists$ good algorithms

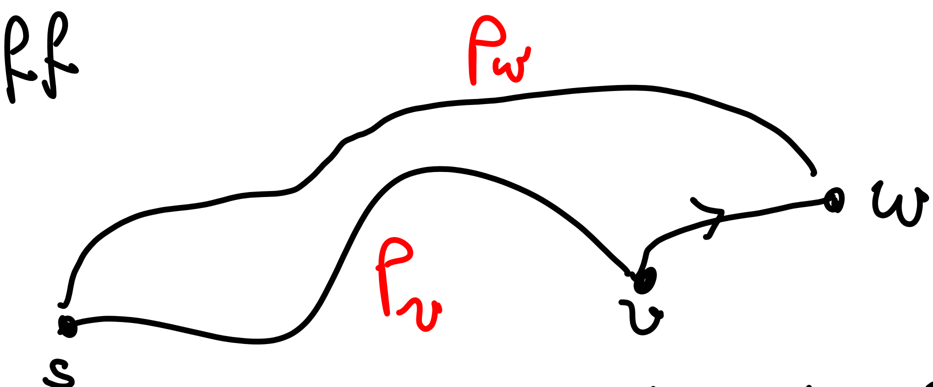
4) If $\exists$ negative cycle then no lower bound on length of shortest walk.

Optimality Criterion

Suppose P_v, $v \in V$ is a collection of walks where P_v is a walk from s to v.

Suppose that $\ell(P_v) = d_v$.

$\{P_v\}$ is a collection of shortest walks iff



(*) $d_w \leq d_v + \ell(v,w), \ \forall v,w$

Proof

Suppose $d_w > d_v + \ell(v,w)$ then

$$\ell(P_v + (v,w)) < \ell(P_w)$$

and so P_w is <u>not</u> a shortest walk.

Conversely, suppose that $d_w \leq d_v + \ell(v,w)$, $\forall v,w$

Must show that P_v is a shortest path, $\forall v$

Let $Q = (s = x_0, x_1, x_2, \ldots, x_k = v)$ be any other path from s to v.

$$d_{x_t} - d_{x_{t-1}} \leq \ell(x_{t-1}, x_t) \quad , \; t=1,2,\ldots,k$$

Add up these inequalities

$$\text{LHS} = d_{x_k} - d_{x_0} = d_v - 0$$

$$\text{RHS} = \ell(Q)$$

Ford's Algorithm

Initialise: $d(s) = 0$

$d(v) = \ell(s,v) \quad \forall v$

∞ if no (s,v) edge

Repeat until *5 holds.

For each edge $e = (v,w)$

check if $d(v) + \ell(v,w) \geq d(w)$

if not $d(w) \Leftarrow d(v) + \ell(v,w)$

Claim: Algorithm finishes in at most n steps.

Let $V(k) = \{ v : \exists$ shortest s to v path with $\leq k$ edges $\}$

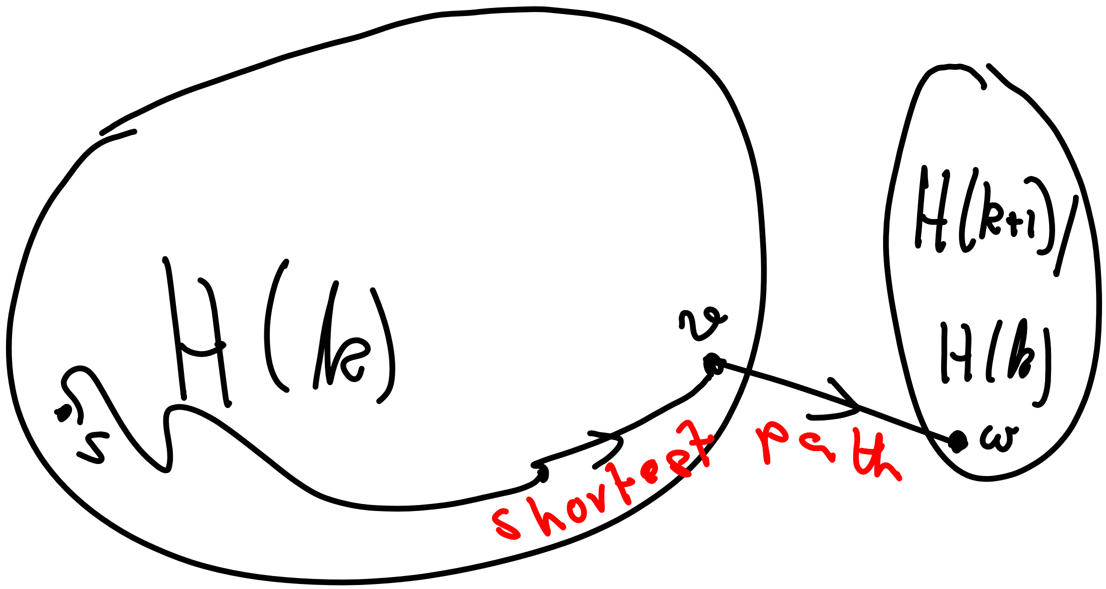
$V(n-1) = V$

After k rounds $d(v)$ is correct for $v \in V(k)$

Proof: Induction on k.

$k=1$ done by initialisation step.

Assume true for k.



After round $k+1$ we have $d(w) \leq d(v) + \ell(v,w)$ is correct.

Need at all times to keep a path of length $d(v)$.