10\14\09

Claim

$\exists$ stable solution $\Leftrightarrow P_A = P_B$

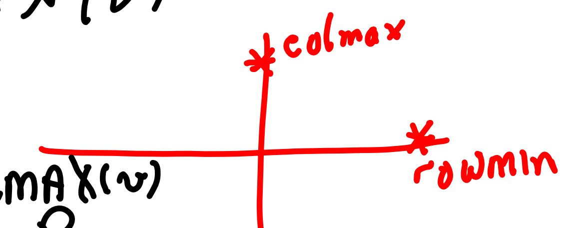
Proof

Suppose u_0, v_0 is stable

(i) $\mathrm{ROWMIN}(u_0) = \mathrm{PAY}(u_0, v_0) = \mathrm{COLMAX}(v_0)$

(ii) $\forall\, u, v \quad \mathrm{ROWMIN}(u) \leq \mathrm{COLMAX}(v)$

(iii) $P_A = \max_u \mathrm{ROWMIN}(u)$
$\geq \mathrm{ROWMIN}(u_0) = \mathrm{COLMAX}(v_0) \geq \min_v \mathrm{COLMAX}(v)$
$= P_B$



i.e. $P_A \geq P_B$. But $P_A \leq P_B$ and so

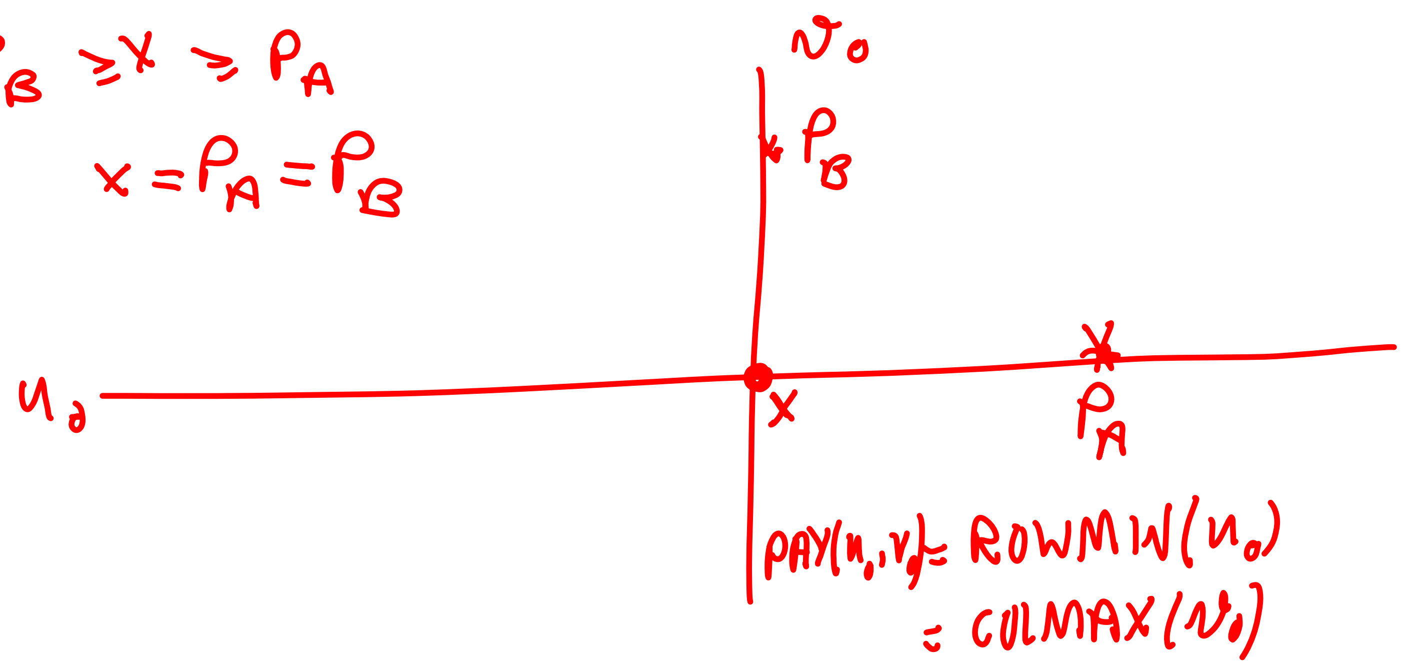
$P_A = P_B$.

Conversely, suppose $P_A = \text{ROWMIN}(u_0)$

$= P_B = \text{COLMAX}(v_0)$

$P_B \geq x \geq P_A$

$x = P_A = P_B$



$S_A = \{ \underline{p} \}$ $\qquad$ $S_B = \{ \underline{q} \}$

$\{ p_1 + \dots + p_m = 1$
$p_1, \dots, p_m \geq 0 \}$

$\{ q_1 + \dots + q_n = 1$
$q_1, \dots, q_n \geq 0 \}$

<u>Claim</u> $\quad P_A = P_B$

$$P_A = \max_{\underline{p}} \min_{\underline{q}} \mathrm{PAY}(\underline{p}, \underline{q})$$

$$= \max_{\underline{p}} \min_{\underline{q}} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, p_i\, q_j$$

$$= \max_{\underline{p}} \min_{\underline{q}} \sum_{j=1}^{n} \underbrace{\left(\sum_{i=1}^{m} a_{ij}\, p_i \right)}_{a_j(\underline{p})} q_j$$

$$= \max_{\underline{p}} \min_{j} a_j(\underline{p})$$

$$\left[\min_{\underline{q}} c_1 q_1 + c_2 q_2 + \cdots + c_n q_n = \min_{j} c_j \right]$$

$$P_A = \max_{\underline{p}} \min_j a_j(\underline{p})$$

$$= \text{maximum } \xi$$

$$\xi \leq a_1(\underline{p})$$
$$\xi \leq a_2(\underline{p})$$
$$\vdots$$
$$\xi \leq a_n(p)$$

$$= \max \xi$$

$$\xi \leq \sum_{i=1}^{m} a_{i1} p_i \qquad p_1 + \cdots + p_m = 1$$
$$\xi \leq \sum_{i=1}^{m} a_{i2} p_i \qquad p_1, p_2, \ldots, p_m \geq 0$$
$$\vdots$$
$$\xi \leq \sum_{i=1}^{m} a_{in} p_i$$

$$P_B = \min_{\underset{\sim}{q}} \max_{\underset{\sim}{p}} PAY(\underset{\sim}{p}, \underset{\sim}{q})$$

$$= \min_{\underset{\sim}{q}} \max_{\underset{\sim}{p}} \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} q_j \right)}_{b_i(\underset{\sim}{q})} p_i$$

$$= \min_{\underset{\sim}{q}} \max_{i} b_i(\underset{\sim}{q})$$

$$= \min \ \eta$$

$$\eta \geq b_1(\underset{\sim}{q})$$

$$\eta \geq b_2(\underset{\sim}{q})$$

$$\vdots$$

$$\eta \geq b_m(q)$$

P_B = minimum η

$$\eta \geq \sum_{j=1}^{n} a_{1j} q_j$$

$$\eta \geq \sum_{j=\text{[redacted]}}^{n} a_{2j} q_j$$

$$\vdots$$

$$\eta \geq \sum_{j=1}^{n} a_{mj} q_j$$

$$q_1 + \dots + q_n = 1$$

$$q_1, \dots, q_n \geq 0$$