9\24\08

$$PAY(p,q) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, p_i\, q_j$$

Claim: Now $P_A = P_B$

$$P_A = \max_p \min_q \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} q_j\right) p_i$$

$$= \max_p \min_q \sum_{j=1}^{n}\underbrace{\left(\sum_{i=1}^{m} a_{ij} p_i\right)}_{X_j(p)} q_j$$

$$= \max_p \min_q \left\{ X_1 q_1 + X_2 q_2 + \cdots + X_n q_n \right\}$$

$$P_A = \max_p \min_q \{ X_1 q_1 + X_2 q_2 + \cdots + X_n q_n \}$$

$$= \max_p \min \{ X_1, X_2, \ldots, X_n \}$$

$$= \max_p \min \left\{ \sum_{i=1}^{m} a_{i1} p_i , \sum_{i=1}^{m} a_{i2} p_i , \ldots \right\}$$

$$= \max \xi$$

$$\xi \le \sum_{i=1}^{m} a_{i1} p_i$$

$$\vdots$$

$$\xi \le \sum_{i=1}^{m} a_{in} p_i$$

$$p_1 + \cdots + p_m = 1$$

$$p_1, p_2, \ldots \ge 0$$

$$\rho_B = \min_q \max_p \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j$$

$$= \min_q \max_p \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} q_j \right)}_{y_i(q)} p_i$$

$$= \min_q \max \{ y_1, y_2, \ldots, y_m \}$$

$$= \min_q \max \left\{ \sum_{j=1}^{n} a_{1j} q_j, \sum_{j=1}^{n} a_{2j} q_j, \ldots \right\}$$

$$= \min \eta \qquad \eta \geq \sum_{j=1}^{n} a_{1j} q_j$$

$$\vdots$$

$$\eta \geq \sum_{j=1}^{n} a_{mj} q_j$$

$$q_1 + \cdots + q_n = 1$$

$$q_1, q_2, \ldots \geq 0.$$

$P_A = \max \xi$

$\xi \leq \sum_{i=1}^{m} a_{i1} p_i$

$\vdots$

$\xi \leq \sum_{i=1}^{m} a_{in} p_i$

$p_1 + \cdots + p_m = 1$

$p_1, p_2, \ldots \geq 0$

$P_B = \min \eta$

$\eta \geq \sum_{j=1}^{n} a_{1j} q_j$

$\vdots$

$\eta \geq \sum_{j=1}^{n} a_{mj} q_j$

$q_1 + \cdots + q_n = 1$

$q_1, q_2, \ldots \geq 0.$

The two linear programs are dual to each other and so $P_A = P_B$.

DUAL

		Dual Variable	
$\max \xi$			minimise z
$\xi - \sum_{i=1}^{m} a_{ij} p_i \leq 0 \ \forall j$		y_j	$-\sum_{i=1}^{m} a_{ij} q_j + z \geq 0 \ \forall i$
$\sum_{i=1}^{m} p_i = 1$		z	$y_1 + \cdots + y_n = 1$
$p_1, \ldots, p_m \geq 0$			$y_1, \ldots, y_n \geq 0$