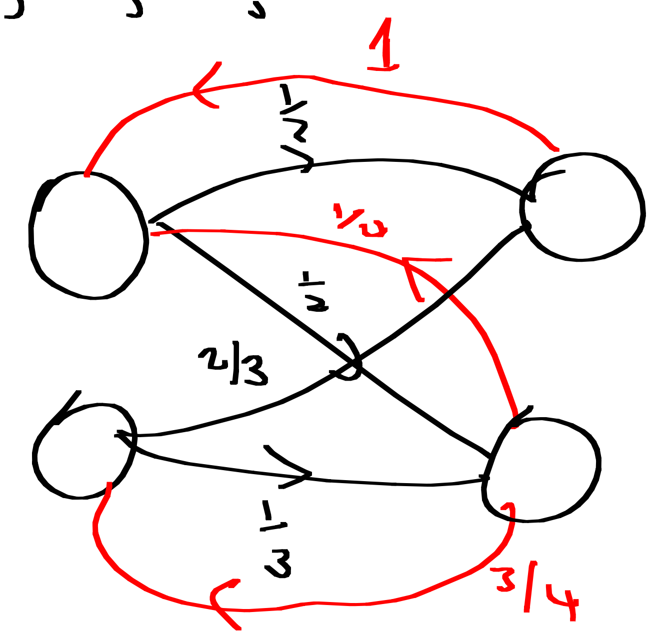
10\13\08

Periodicity

Period of a state is n if starting at i, can only return at $n, 2n, 3n,$



$n = 2$

Chain is Ergodic if all states have period 1.

Steady State π

Suppose chain is (i) finite
(ii) ergodic
(iii) irreducible

Then

$$\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j$$

independent of i

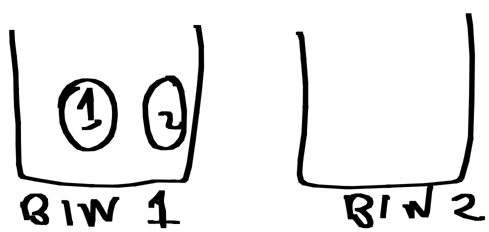
$$P^n \to \begin{bmatrix} \pi_1 & \pi_2 & \cdots & \pi_N \\ \pi_1 & \pi_2 & \cdots & \pi_N \\ & \vdots & & \\ \pi_1 & \pi_2 & \cdots & \pi_N \end{bmatrix}$$

π is the unique solution to

(i) $\pi P = \pi$

(ii) $\pi_1 + \pi_2 + \cdots + \pi_N = 1$

Perron-Frobenius



States of chain are $0, 1, 2$ = # of balls in bin 1.

Move: Choose random bin.
Choose ball, put it into random bin.

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$\frac{3}{4}\pi_1 + \frac{1}{4}\pi_2 = \pi_1$$

$$\frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = \pi_2$$

$$\frac{1}{4}\pi_2 + \frac{3}{4}\pi_3 = \pi_3$$ Redundant

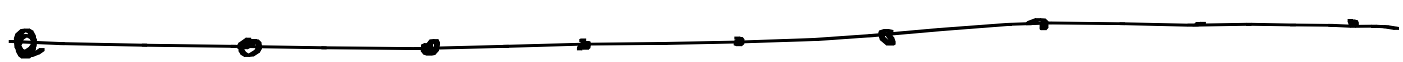
$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

Costs: Suppose C_j = expected cost of being in state j for one period.

Long run expected cost

$$\approx \sum_{j=1}^{N} \pi_j C_j$$



D = uniform over $\{0,1,2,3,4\}$

$c_j = j$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{1}{5} & 0 & 0 \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \end{array}$$

$$\frac{2}{5}\pi_0 + \frac{4}{5}\pi_1 + \frac{3}{5}\pi_2 + \frac{3}{5}\pi_3 = \pi_0 = \frac{64}{125}$$

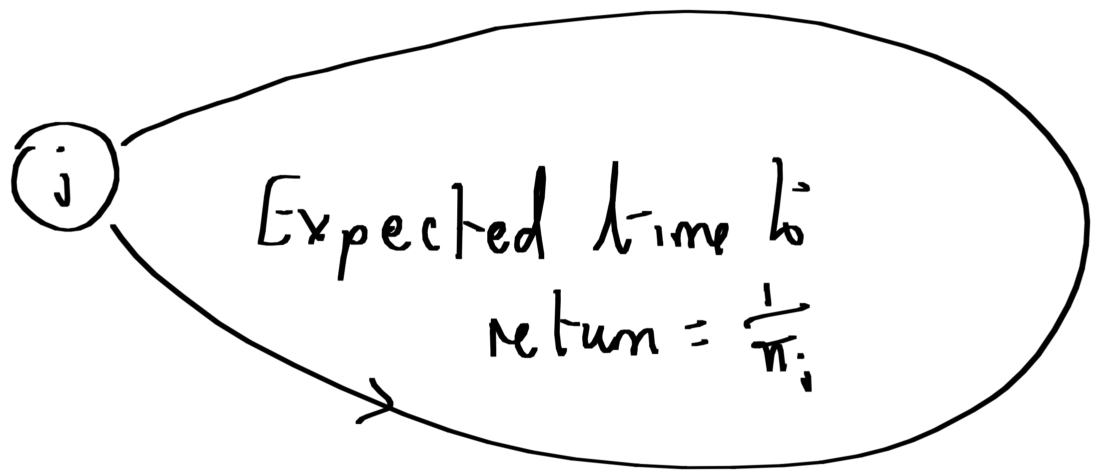
$$\frac{1}{5}\pi_0 + \frac{1}{5}\pi_1 + \frac{1}{5}\pi_2 + \frac{1}{5}\pi_3 = \pi_1 = \frac{1}{5}$$

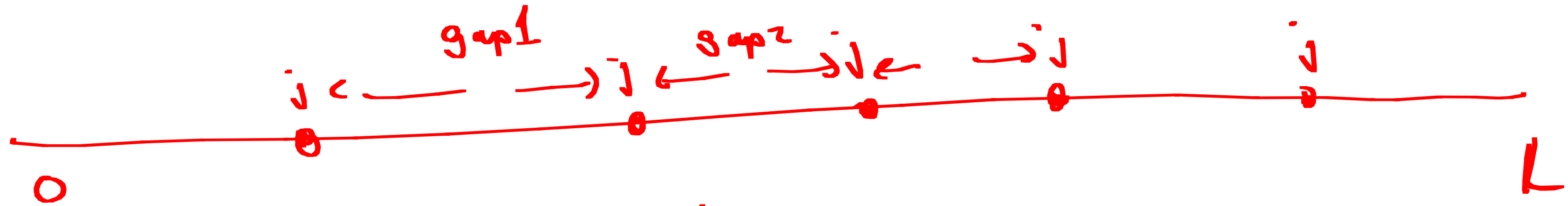
$$\frac{1}{5}\pi_0 + \frac{1}{5}\pi_2 + \frac{1}{5}\pi_3 = \pi_2 = \frac{4}{25}$$

$$\frac{1}{5}\pi_0 + \frac{1}{5}\pi_3 = \pi_3 = \frac{16}{125}$$

Long Run Expected Cost =

$$\frac{1}{5} + \frac{8}{25} + \frac{48}{125}$$





$\#$ of j's $\approx L\pi_j$

$\sum$ gaps $= L$

Average gap length $\approx \frac{L}{L\pi_j}$