10\10\08

Markov Chains

Stochastic Process

$X_0, X_1, X_2, \ldots, X_t, \ldots$

is a sequence of random variables.

e.g. Weather : $X_t = \begin{cases} 1 & \text{Sunny on day } t \\ -1 & \text{Raining on day } t \end{cases}$

Markov chain is a special sort of stochastic process.

Markov Property:

$$\Pr[X_{n+1}=\omega \mid X_0=\omega_0, X_1=\omega_1, \dots, X_n=\omega_n]$$

$$= \Pr[X_{n+1}=\omega \mid X_n=\omega_n]$$

$$= \Pr[X_1=\omega \mid X_0=\omega_n]$$

A stochastic process with the Markov property is called a Markov Chain

Inventory Example

X_i = amount in stock at start of period i

$D_0, D_1, \ldots D_i, \ldots$ = demand in period i

Ordering policy: If $X_i = 0$, order 3
else order 0

$$D_i = \begin{array}{cc} 0 & 1/5 \\ 1 & 1/5 \\ 2 & 1/5 \\ 3 & 1/5 \\ 4 & 1/5 \end{array} \quad \text{uniform}$$

$P[X_{i+1} = 0 \mid X_i = 0] = 2/5$

$P[X_{i+1} = 0 \mid X_i = 1] = \frac{4}{5}$

Transition Probabilities

$$P_{ij} = P[X_{n+1} = j \mid X_n = i]$$

$$P = \| P_{ij} \| = \text{transition matrix}$$

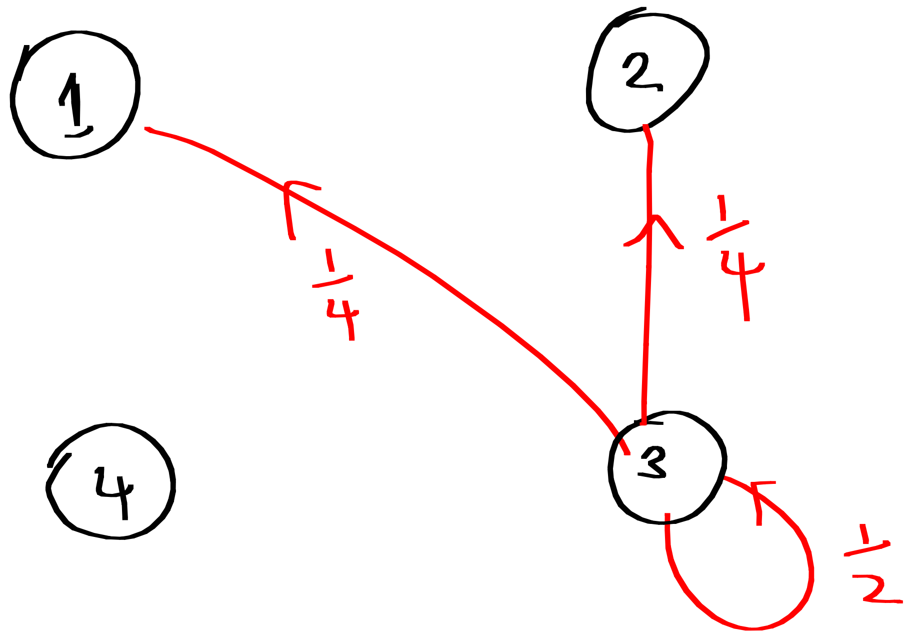
Gambling Example

$$\begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Win 1 with prob p

Lose 2 " " $1-p$

Transition Diagram



4 states
$\equiv$
values of X

$$
3\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \end{bmatrix}
$$

Start of chain:

$$\Pr[X_0 = i] = p_i^{(0)}$$

$$p_j^{(1)} = \Pr[X_1 = j] = \sum_i p_i^{(0)} P_{ij}$$

$$p^{(1)} = p^{(0)} P; \quad p^{(2)} = p^{(1)} P = p^{(0)} P^2$$

$$p_j^{(t)} = \Pr[X_t = j]$$

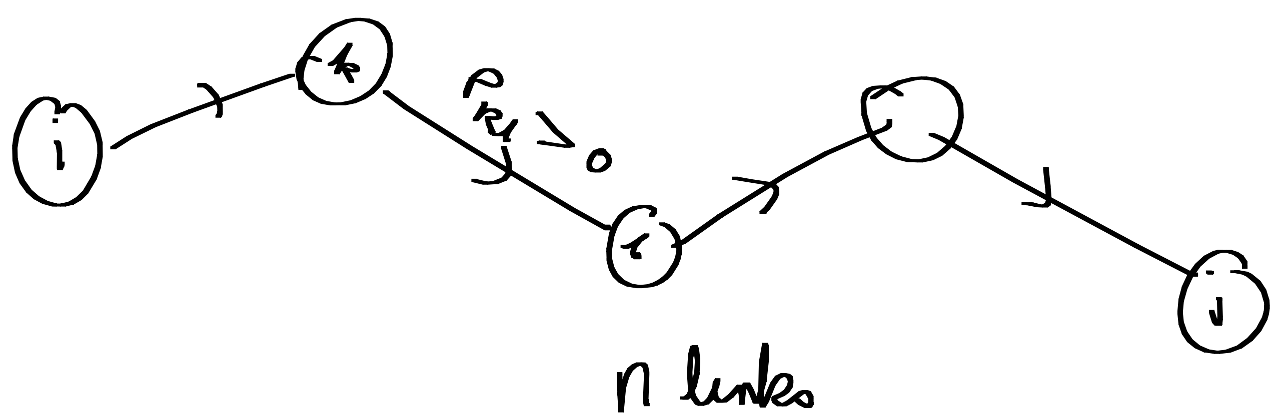
$$p^{(t)} = p^{(0)} P^t$$

Classification of States

j is accessible from i

if $\exists n : P_{ij}^{(n)} > 0$

$P[X_n = j \mid X_0 = i]$



$$D = \text{digraph}\,(\underbrace{\Omega}_{\text{states}},\ \underbrace{A}_{\substack{(i,j)\in A \\ \Leftrightarrow P_{ij}>0}})$$

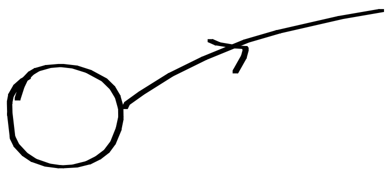
j is accesible from i if $\exists$ a path from i to j in D

$$\Omega = \Omega_1 \cup \Omega_2 \cup \dots \cup \Omega_k$$

classes (strong components of D)

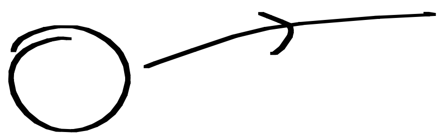
Irreducible if $k = 1$

Transient State



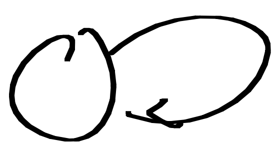
$P_r(\text{return}) < 1$
eventually

Recurrent State



$P_r(\text{return}) = 1$
eventually

Absorbing State



$P_{i,i} = 1$

States of a class
are either
all transient
or
all recurrent