Department of Mathematical Sciences
**CARNEGIE MELLON UNIVERSITY**

**OPERATIONS RESEARCH II 21-393**

Homework 2: Due Monday September 24.

**Q1** Consider an electronic system consisting of four components each of which must work for the system to function. The reliability of the system can be improved by installing several parallel units in one or more of the components. The following table gives the probability that the respective components will function if they consist of one, two or three parallel units.

| Number of units | Component 1 | Component 2 | Component 3 | Component 4 |
|---|---|---|---|---|
| 1 | 0.4 | 0.6 | 0.7 | 0.5 |
| 2 | 0.6 | 0.7 | 0.8 | 0.7 |
| 3 | 0.8 | 0.8 | 0.9 | 0.9 |

The probability that the system will function is the product of the probabilities that the respective components will function. The cost (in hundred's of dollars) of installing one, two or three parallel units in the respective compoonents is given by the following table.

| Number of units | Component 1 | Component 2 | Component 3 | Component 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 1 | 1 |
| 2 | 2 | 4 | 3 | 3 |
| 3 | 3 | 5 | 4 | 4 |

Because of budget limitations, a maximum of \$1000 can be spent. use Dynamic Programming to determine how many parallel units should be installed in each of the four components in order to maximise the probability that the system will function.

**Solution:** Let $p_i(x)$ be the probability that component $i$ functions if there are $x$ parallel units placed in it and let $c_r(x)$ be the cost of putting $x$ parallel units into component $i$. Let $\phi_r(w)$ be the maximum reliability of a system of components $r, r+1, \ldots, 4$ if one has enough money to put in $w$ parallel components altogether. We wish to compute $\phi_1(10)$. In general

$$\begin{aligned} \phi_4(w) &= p_4(w) \\ \phi_r &= \max_x (p_r(x)\phi_{r+1}(w - c_r(x))) \end{aligned}$$

| $w$ | $\phi_1$ | $x_1$ | $\phi_2$ | $x_2$ | $\phi_3$ | $x_3$ | $\phi_4$ | $x_4$ |
|---|---|---|---|---|---|---|---|---|
| 0 | | | 0 | | 0 | | 0 | |
| 1 | | | 0 | | 0 | | 0 | |
| 2 | | | 0 | | .35 | 1 | .5 | 1 |
| 3 | | | 0 | | .35 | 1 | .7 | 2 |
| 4 | | | 0 | | .49 | 1 | .9 | 3 |
| 5 | | | .210 | 1 | .63 | 1 | .9 | 3 |
| 6 | | | .294 | 1 | .63 | 1 | .9 | 3 |
| 7 | | | .378 | 1 | .72 | 2 | .9 | 3 |
| 8 | | | .378 | 1 | .81 | 3 | .9 | 3 |
| 9 | | | .441 | 2 | .81 | 3 | .9 | 3 |
| 10 | .3024 | 3 | .504 | 3 | .81 | 3 | .9 | 3 |

Solution: $x_1 = 3, x_2 = 1, x_3 = 1, x_4 = 3$. Maximum = .3024.

**Q2** A system can be in 3 states 1,2,3 and the cost of moving from state $i$ to state $j$ in one period is $c(i, j)$, where the $c(i, j)$ are given in the matrix below. The one period discount factor $\alpha$ is 1/2.

The aim is to find a policy which simultaneously minimises the discounted cost of operating from any starting state. Start with the policy

$$\pi(1) = 1, \pi(2) = 1, \pi(3) = 2.$$

Find an optimal policy.

The matrix of costs is

$$\begin{bmatrix} 8 & 2 & 3 \\ 5 & 2 & 8 \\ 1 & 7 & 2 \end{bmatrix}$$

**Solution:** Start with an initial solution of $\pi(1) = 1, \pi(2) = 1, \pi(3) = 2$.

Then we solve

$$\begin{aligned} y_1 &= 8 + \frac{y_1}{2} \\ y_2 &= 5 + \frac{y_1}{2} \\ y_3 &= 7 + \frac{y_2}{2} \end{aligned}$$

Solving these equations gives $y_1 = 16, y_2 = 13, y_3 = 27/2$.
We now check for optimality:
Check $y_1$:

$$8 + \frac{y_1}{2} = 16$$

$$2 + \frac{y_2}{2} = \frac{17}{2} \quad \star$$

$$3 + \frac{y_3}{2} = \frac{39}{4}$$

Check $y_2$:

$$5 + \frac{y_1}{2} = 13$$

$$2 + \frac{y_2}{2} = \frac{17}{2} \quad \star$$

$$8 + \frac{y_3}{2} = \frac{59}{4}$$

Check $y_3$:

$$1 + \frac{y_1}{2} = 9$$

$$7 + \frac{y_2}{2} = \frac{27}{2}$$

$$2 + \frac{y_3}{2} = \frac{35}{4} \quad \star$$

Our new policy is $\pi(1) = 2, \pi(2) = 2, \pi(3) = 3$.

Then we solve

$$
\begin{aligned}
y_1 &= 2 + \frac{y_2}{2} \\
y_2 &= 2 + \frac{y_2}{2} \\
y_3 &= 2 + \frac{y_3}{2}
\end{aligned}
$$

Solving these equations gives $y_1 = 4, y_2 = 4, y_3 = 4$.
We now check for optimality:
Check $y_1$:

$$
\begin{aligned}
8 &+ \frac{y_1}{2} = 10 \\
2 &+ \frac{y_2}{2} = 4 \quad \star \\
3 &+ \frac{y_3}{2} = 5
\end{aligned}
$$

Check $y_2$:

$$
\begin{aligned}
5 &+ \frac{y_1}{2} = 7 \\
2 &+ \frac{y_2}{2} = 4 \quad \star \\
8 &+ \frac{y_3}{2} = 10
\end{aligned}
$$

Check $y_3$:

$$
\begin{aligned}
1 &+ \frac{y_1}{2} = 3 \quad \star \\
7 &+ \frac{y_2}{2} = 9 \\
2 &+ \frac{y_3}{2} = 4
\end{aligned}
$$

Our new policy is $\pi(1) = 2, \pi(2) = 2, \pi(3) = 1$.

Then we solve

$$\begin{aligned} y_1 &= 2 + \frac{y_2}{2} \\ y_2 &= 2 + \frac{y_2}{2} \\ y_3 &= 1 + \frac{y_1}{2} \end{aligned}$$

Solving these equations gives $y_1 = 4, y_2 = 4, y_3 = 3$.
We now check for optimality:
Check $y_1$:

$$\begin{aligned} 7 + \frac{y_1}{2} &= 9 \\ 2 + \frac{y_2}{2} &= 4 \quad \star \\ 3 + \frac{y_3}{2} &= \tfrac{13}{2} \end{aligned}$$

Check $y_2$:

$$\begin{aligned} 5 + \frac{y_1}{2} &= 7 \\ 2 + \frac{y_2}{2} &= 4 \quad \star \\ 7 + \frac{y_3}{2} &= \tfrac{17}{2} \end{aligned}$$

Check $y_3$:

$$\begin{aligned} 1 + \frac{y_1}{2} &= 3 \quad \star \\ 6 + \frac{y_2}{2} &= 8 \\ 2 + \frac{y_3}{2} &= \tfrac{7}{2} \end{aligned}$$

So the current policy is optimal.
**Q3** You are an oil trader and the price of oil fluctuates in the following way:
If the price per barrel in period $i$ is $p_i$ dollars then

$$p_{i+1} = \begin{cases} (1+\varepsilon)p_i & Probability\ p \\ (1-\varepsilon)p_i & Probability\ 1-p \end{cases}$$

The current price is one hundred dollars per barrel. Give a dynamic programming formulation to compute the maximum price per barrel one should pay now for the following option: Purchase oil at price $c$ dollars per barrel, at any time $t = 1, 2, \ldots, n$.
Explanation: Here one is paying for the (hoped for) opportunity to purchase the oil cheaply in the future.
**Solution:** Observe first that the price per barrel of oil at time $t$ takes values in the set $\{100(1+\varepsilon)^a(1-\varepsilon)^{t-1-a} : \; a = 0, 1, \ldots, t-1\}$.
Let $f_t(a)$ be the maximum expected value of the option at time $t$ if the current discount price is $100(1+\varepsilon)^a(1-\varepsilon)^{t-1-a}$. Then

$$f_t(a) = \max\{100(1+\varepsilon)^a(1-\varepsilon)^{t-1-a} - c, \, pf_{t+1}(a+1) + (1-p)f_{t+1}(a)\}$$

and $f_{n+1}(a) = 0$ for all $a$.
You should pay no more than $f(t, 0)$.