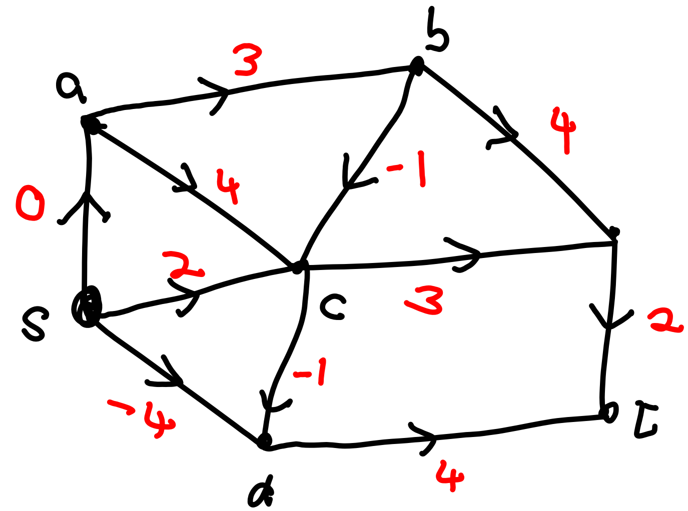
10\12\2007

Shortest Path Problem.

Digraph = (N, A)

nodes arcs



Path : — no vertex is repeated

Walk : any sequence of arcs, connected.

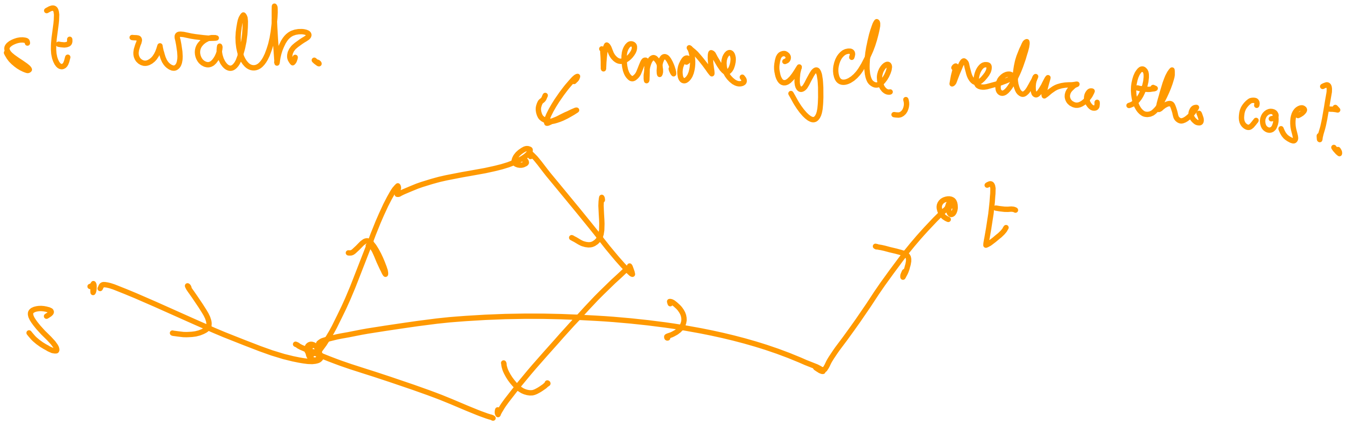
Problem: Find minimum length path s to t.

Negative Cycles

If D has a cycle C with $\ell(C) < 0$ then no gurantee of an efficient algorithm.

Assume: No negative cycles.

Shortest path from $s \longrightarrow t$ is also the shortest walk.



We can look for a shortest walk.

Suppose $P_j : j \in N$ is a collection of paths where P_j is a path from s to j

$d_j = \ell(P_j)$.

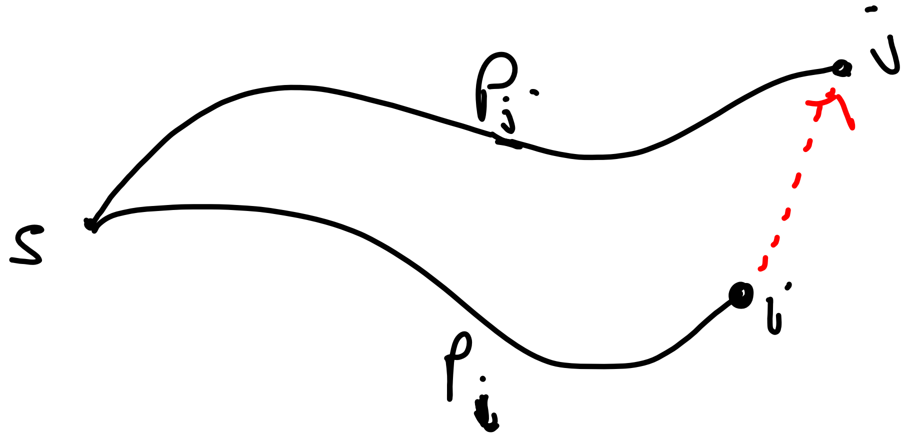
<u>Thm</u>

These are all shortest paths iff

$$d_j \leq d_i + \ell(i,j), \quad \forall (i,j) \in A.$$

Proof

Only If: $d_j > d_i + l(i,j)$

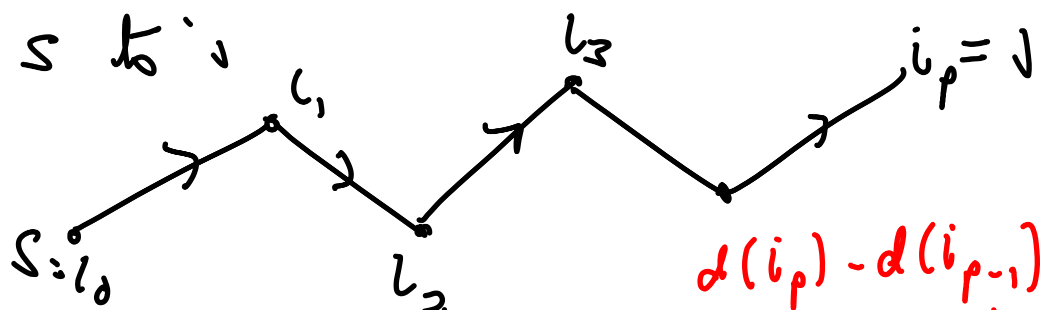


lower is a shorter walk from s to j

If:

Any path s to j

P



$d(i_p) - d(i_{p-1}) \le l(i_{p-1}, i_p)$

$d(i_{p-1}) - d(i_{p-2}) \le l(i_{p-2}, i_{p-1})$

$\vdots$

$d(i_1) - d(s) \le d(s, i_1)$

ADD

$l(P) \ge d(j)$

Ford's Algorithm

$d(x)$ is always the length of a walk

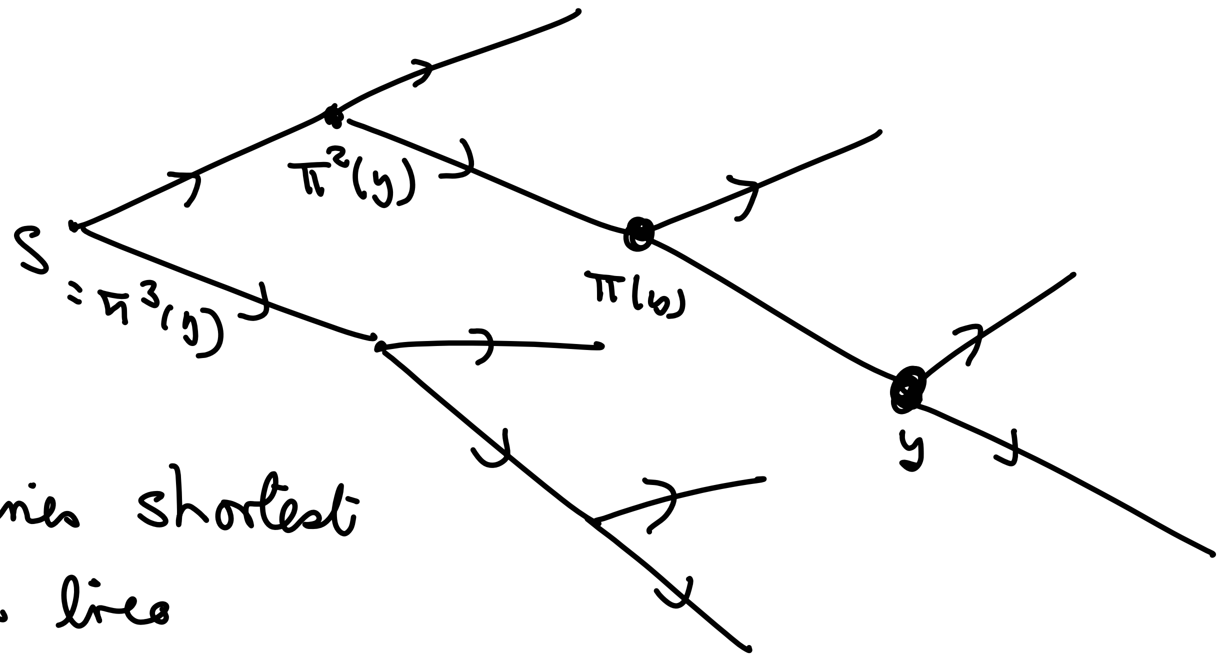
$A = \{u_1, u_2, \dots, u_m\}$ $\quad u_i = \{x_i, y_i\}$

Initialise: $d(s)=0$; $d(v)=\infty$ for $v \neq s$;

Repeat Until Optimality Condition Holds

$$\left\{ \begin{array}{l} \text{for } i = 1, 2, \dots, m \\ \quad \text{if } \quad d(y_i) > d(x_i) + l(x_i, y_i) \\ \quad \text{then } \quad d(y_i) = d(x_i) + l(x_i, y_i) \; ; \; \pi(y_i) = x_i \end{array} \right.$$

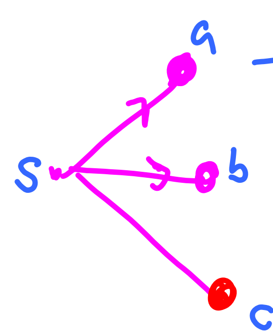
Stop if you go through loop without a change.



π defines shortest path tree

Ford's algorithm terminates with optimum paths.

$d(s)$ is always correct



— shortest path to a of length 1.

$S(k) = \{ v : \exists$ shortest path with k edges $\}$

$d(v)$ is correct after k rounds : Induction